



Extremal $GI/GI/1$ Queues Given First Two Moments

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Motivation

- ▶ Queueing Network Analyzer in Whitt (1983).
 - ▶ Evaluate Limits of Queueing Approximations
 - ▶ Propose Upper/Lower Bounds for Queueing Characteristics
 - ▶ Extremal Queueing Models
- ▶ Question: How to Explore Extremal Queues?



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- ▶ Queueing Network Analyzer in Whitt (1983).
- ▶ Evaluate Quality of Queueing Approximations.
- ▶ Propose a new Upper/Lower Bound for Queueing Characteristics in *Extremal Queueing Models*.

Research Question: How to Explore Extremal Queues?



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- ▶ Queueing Network Analyzer in Whitt (1983).
- ▶ Evaluate Quality of Queueing Approximations.
- ▶ Propose Tight Upper & Lower Bounds for Queueing Characteristics (*Extremal Queueing Models*).

Research Question: How to Explore Extremal Queues?



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- ▶ Propose Tight Upper & Lower Bounds for Queueing Characteristics (*Extremal Queueing Models*).

Research Question: How to Explore Extremal Queues?



Basic Settings

GI/GI/1 Model:

- ▶ **unlimited** waiting room and **FCFS** discipline.

▶ mean inter-arrival time and service time ($m_1 = 1, s_1 = \rho$).

▶ second moment $m_2, m_2 = (1 + \text{var}(X)) = (1 + s_1^2)s_1^2$.

▶ inter-arrival time U distributed as $U \in [0, M_1]$

▶ service time V distributed as $V \in [0, M_2]$

▶ $\rho < 1$ (stability)

▶ see e.g. [Gross and Harris (1967), Ott(1987), Whitt (1984b),

Chen and Wang (2005), Whitt (2005)]

▶ see e.g. [Gajda and Székely (2007), Gupta and Osogami (2007),

Chen and Whitt (2009), Raymond (2011)]



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▶ second moment $m_2 = (1 + \frac{\rho^2}{s_1^2})s_1^2$.

▶ inter-arrival time I distributed as $I \sim \text{Exp}(\lambda)$ and

▶ service time V distributed as $V \sim [0, M]$.

▶ $\rho < 1$ is assumed.

▶ See e.g. Whitt (1996), Whitt (1997), Ott (1987), Whitt (1984b),

Whitt (1998), Wang (2005).

▶ See e.g. Gajani (2007), Gupta and Osogami (2007),

Whitt (2007), Raymond (2011).



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- ▶ mean inter-arrival time and service time ($m_1 = 1, s_1 = \rho$).
- ▶ second moments $m_2 = (1 + c_a^2)m_1^2, s_2 = (1 + c_s^2)s_1^2$.

▶ inter-arrival times $\{t_i\}$ distributed as $G(m_1, m_2)$ with $\rho < 1$

▶ service times $\{s_i\}$ distributed as $G(s_1, s_2)$ with $\rho < 1$

▶ $\rho = m_1 s_1$

▶ $\rho < 1$ (e.g., Whitt (1984), Whitt (1987), Ott (1987), Whitt (1984b),

Whitt (1984c), Wang (2005),

Whitt (2006), Whitt (2007), Gajani (2007), Gupta and Osogami (2007),

Whitt (2008), Whitt (2009), Raymond (2011).



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- ▶ inter-arrival time U distributed as F over $[0, M_a]$.
- ▶ service time V distributed as G over $[0, M_s]$.

Related Literature

Chen (2006), Chen and Wang (2005), Ott (1987), Wang (1984), Wang (2005), Wang and Zhang (2007), Chen and Osogami (2007), Chen and Raymond (2011).

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Related Literature

Chen (1997), Gans et al. (1997), Ott (1987), Wolff (1981),
 Gans et al. (2003), Wolff and Zhang (2005),
 Gans et al. (2006), Gans et al. (2007), Gans and Osoodati
 (2008), Gans et al. (2009), Raymond (2011).



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Related Literature:

Chen (2004), Gans et al. (2003), Ott (1987), Whitt (1984),
 Whitt (1993), Whitt (2005), Wang (2005),
 Dhillon and Mahajan (2007), Gupta and Osogami
 (2010), Nagarajan and Raymond (2013).



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Related Literature:

- ▶ Kingman (1962), Daley (1977), Ott(1987), Whitt (1984b), Daley (1992), Wolff and Wang (2003).

Boylan and Neuman (1967), Garcia and Osogami (2011), Bergamini and Raymond (2013).



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- ▶ inter-arrival time U distributed as F over $[0, M_a]$.
- ▶ service time V distributed as G over $[0, \rho M_s]$.

Related Literature:

- ▶ Kingman (1962), Daley (1977), Ott(1987), Whitt (1984b), Daley (1992), Wolff and Wang (2003).
- ▶ Bertsimas and Natarajan (2007), Gupta and Osogami (2011), Osogami and Raymond (2013).



Basic Settings

- $\mathcal{P}_2(M)$: cdf's with support in the interval $[0, mM]$ given mean m and second moment $(c^2 + 1)m^2$.

► $\mathcal{P}_{2,k}(M)$: $\{F \in \mathcal{P}_2(M) : \text{supp}(F) \subseteq [0, mM], F(0) = 1/k\}$

► Steady-state Waiting Time:

$$W = \sum_{i=1}^{\infty} W_i, \quad W_i = V_i + Q_i$$

with $W_0 = 0$

► $\mathcal{P}_{\alpha,2}(M_s) \times \mathcal{P}_{\alpha,2}(M_s) \rightarrow \mathbb{R}$, $(F, G) \mapsto \mathcal{E}(F, G)$

where $0 < \alpha < 1$ and

$$\mathcal{E}(F, G) \equiv E(V(F, G))$$



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▶ Steady-state waiting time:

$$W = \sum_{i=1}^{\infty} W_i$$

with $W_0 = 0$

$$W_i = \int_0^{\infty} \int_0^{\infty} \mathbb{1}_{\{x > y\}} dP_{i-1}(M_s) \times dP_i(M_s) \rightarrow \mathbb{R},$$

where $0 < \rho < 1$ and

$$P_i(M_s) \equiv E[V(F, Q)]$$



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- ▶ $\mathcal{P}_{2,k}(M)$: cdf's with finite k support in $\mathcal{P}_2(M)$.
- ▶ **Steady-state** Waiting Time:

$$W \stackrel{d}{=} [W + V - U]^+,$$

with $W_0 = 0$.

$$w : \mathcal{P}_{a,2}(M_a) \times \mathcal{P}_{s,2}(M_s) \rightarrow \mathbb{R},$$

where $0 < \rho < 1$ and

$$w(F, G) \equiv E[W(F, G)].$$



Background

- Pollaczek-Khintchine formula:

$$E[W(M, G)] = \frac{\tau\rho(1 + c_s^2)}{2(1 - \rho)} = \frac{\rho^2(1 + c_s^2)}{2(1 - \rho)}.$$

- Kingman's G/G/1 bound:

$$E[W(M, G)] \leq \frac{E[M^2] + E[G^2]}{2(1 - \rho)}.$$

- Daley (1977) bound:

$$E[W(M, G)] \leq \frac{\rho^2((c_s^2 - \rho)c_s^2/4 + 1 + c_s^2)}{2(1 - \rho)}.$$



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- ▶ Kingman (1962) bound:

$$E[W(F, G)] \leq \frac{\rho^2([c_a^2/\rho^2] + c_s^2)}{2(1 - \rho)}.$$

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- ▶ Daley (1977) bound:

$$E[W(F, G)] \leq \frac{\rho^2([(2 - \rho)c_a^2/\rho] + c_s^2)}{2(1 - \rho)}.$$



Objectives

We want to answer a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any F, G , what is the extremal inter-arrival time dist F^* attaining the UB of $\text{EJW}(F, G)$?
- ▶ Given any F, G , what is the extremal service time dist G^* attaining the UB of $\text{EJW}(F, G)$?
- ▶ Given any F, G , what is the extremal F^*, G^* leading to overall UB of $\text{EJW}(F, G)$?



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We want to answer a **Long Standing Open Problem** in Queueing Theory.

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- ▶ What are the extremal F^* and G^* leading to **overall** UB of $\mathbb{E}[W(F, G)]$?



Extremal Distributions

Two-point distributions (**one parameter family**):

- $F \in \mathcal{P}_{a,2,2}(M_a)$: $c_a^2/(c_a^2 + (b_a - 1)^2)$ at b_a ,
 $(b_a - 1)^2/(c_a^2 + (b_a - 1)^2)$ at $1 - c_a^2/(b_a - 1)$
 where $1 + c_a^2 \leq b_a \leq M_a$.

► $G \in \mathcal{P}_{s,2,2}(M_s)$: $(c_s^2 + (b_s - 1)^2)/(c_s^2 + (b_s - 1)^2 + (b_s - 1)^3)$ at b_s ,
 $(b_s - 1)^3/(c_s^2 + (b_s - 1)^2 + (b_s - 1)^3)$ at $1 - c_s^2/(b_s - 1)$
 where $1 + c_s^2 \leq b_s \leq M_s$.

► $F = F_u$ for $b_a = 1 + c_a^2$ and $G = F_u$ when $b_a = M_a$
 and $G = G_u$ when $b_s = 1 + c_s^2$ and $G = G_u$ when $b_s = M_s$.



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• $F = F_0$ for $b_a = 1 + c_a^2$ and $F = F_u$ when $b_a = M_u$
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- ▶ $G = G_0$ for $b_s = 1 + c_s^2$ and $G = G_u$ when $b_s = M_s$.



Main Results for Extremal Queues

- ▶ (a) Given any parameter vector $(1, c_a^2, \rho, c_s^2)$ over bounded support $[0, M_a]$ and $[0, \rho M_s]$, the pair (F_0, G_u) attains the tight UB of the steady-state mean $E[W(F, G)]$, while a pair $(F_0, G_{u,n})$ attains the tight UB of the transient mean $E[W_n(F, G)]$, where $G_{u,n}$ is a two-point distribution with $G_{u,n} \Rightarrow G_u$ as $n \rightarrow \infty$.



Main Results for Extremal Queues

- ▶ (b) For the unbounded interval of support $[0, \infty)$, the tight UB of $E[W]$ is not attained directly, but is obtained asymptotically in the limit as $M_s \rightarrow \infty$ in part (a).
- ▶ (c) The mean $E[W(F_0, G_u)]$ does not approach the mean in the associated extremal $F_0/D/1$ queue as $M_s \rightarrow \infty$, yet approach to $\lim_{M_s \rightarrow \infty} E[(W(F_0, G_u)) (\mathbb{E}[W(F_0, G_{u^*})])]$.



Main Results for Upper Bound

Overall Upper Bound Inequalities:

$$\begin{aligned}
 \mathbb{E}[W(F, G)] &\leq \mathbb{E}[W(F_0, G_{ac})] \text{ (Tight UB)} \\
 &\leq \frac{2(1-\rho)\rho/(1-\rho)^2 + \rho^2}{2(1-\rho)} \text{ (2-Approx)} \\
 &\leq \frac{\rho^2}{2(1-\rho)} \text{ (2-Approx)} \\
 &\leq \frac{\rho^2}{2(1-\rho)} \text{ (Kingman(1962))} \\
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 \end{aligned}$$



Main Results for Upper Bound

Overall Upper Bound Inequalities:

$$\mathbb{E}[W(F, G)] \leq \mathbb{E}[W(F_0, G_{u^*})] (\text{Tight UB})$$

$$\frac{2(1-\rho)\rho/(1-\rho^2)}{2(1-\rho)/(1-\rho^2)} \approx 1 \quad (\text{Approx})$$

$$\frac{\rho^2(1-\rho^2)/(1-\rho^2)}{2(1-\rho)/(1-\rho^2)} \approx \frac{\rho^2}{2(1-\rho)}$$

$$\frac{\rho^2}{2(1-\rho)} \approx \frac{\rho^2}{2} \quad (\text{Kingman(1962)})$$

$$\frac{\rho^2}{2(1-\rho)} \approx \rho(-1-\delta)/4$$



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Overall Upper Bound Inequalities:

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 &< \frac{\rho^2([(2-\rho)c_a^2/\rho] + c_s^2)}{2(1-\rho)} \text{ (Daley(1977))} \\
 &< \frac{\rho^2(c_a^2 + c_s^2)}{2(1-\rho)} \text{ (Kingman(1976))}
 \end{aligned}$$

where $\delta = 1 - \exp(-(1-\rho)/\rho)$.



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 \end{aligned}$$

where $\delta \in (0, 1)$ and $\rho = \exp(-(1-\delta)/\rho)$.



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where $\delta \in (0, 1)$ and $\delta = \exp(-(1-\delta)/\rho)$.



Main Results for Upper Bound

Table: A comparison of the bounds and approximations for the steady-state mean $\mathbb{E}[W]$ as a function of ρ for the case $c_a^2 = c_s^2 = 4.0$.

ρ	Tight LB	HTA	Tight UB	UB Approx	δ	MRE	Daley	Kingman
0.10	0.00	0.044	0.422	0.422	0.000	0.003%	0.44	2.24
0.20	0.00	0.200	0.904	0.906	0.007	0.19%	1.00	2.60
0.30	0.00	0.514	1.499	1.51	0.041	0.60%	1.71	3.11
0.40	0.00	1.07	2.304	2.33	0.107	0.94%	2.67	3.87
0.50	0.25	2.00	3.470	3.51	0.203	1.15%	4.00	5.00
0.60	1.00	3.60	5.295	5.35	0.324	1.07%	6.00	6.80
0.70	2.42	6.53	8.441	8.52	0.467	0.93%	9.33	9.93
0.80	5.50	12.80	14.92	15.02	0.629	0.67%	16.00	16.40
0.90	15.25	32.40	34.72	34.84	0.807	0.35%	36.00	36.20
0.95	35.13	72.20	74.62	74.76	0.902	0.18%	76.00	76.10
0.98	95.05	192.1	194.6	194.7	0.960	0.07%	196.0	196.0
0.99	195.0	392.0	394.5	394.7	0.980	0.04%	396.0	396.0



Main Reduction Theorem

Theorem

(reduction to a three-point distribution) Consider the class of $GI/GI/1$ queues with cdf $F \in \mathcal{P}_{a,2}$ with cdf $G \in \mathcal{P}_{s,2}$ where $0 < \rho < 1$, the following suprema are attained as indicated:

(a) For any specified $G \in \mathcal{P}_{s,2}$, there exists $F^(G) \in \mathcal{P}_{a,2}(M_0)$ such that*

$$\begin{aligned} w_a^*(G) &\equiv \sup \{w(F, G) : F \in \mathcal{P}_{a,2}(M_0)\} \\ &= \sup \{w(F, G) : F \in \mathcal{P}_{a,2,p}(M_0)\} = w(F^*(G), G). \end{aligned}$$



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(a) For any specified $G \in \mathcal{P}_{s,2}$, there exists $F^(G) \in \mathcal{P}_{a,2,3}(M_a)$ such that*

$$\begin{aligned} w_a^\uparrow(G) &\equiv \sup \{w(F, G) : F \in \mathcal{P}_{a,2}(M_a)\} \\ &= \sup \{w(F, G) : F \in \mathcal{P}_{a,2,3}(M_a)\} = w(F^*(G), G). \end{aligned}$$



Main Reduction Theorem

Theorem

(continued)

(b) For any specified $F \in \mathcal{P}_{a,2}$, there exists $G^*(F) \in \mathcal{P}_{s,2,3}(M_s)$ such that

$$\begin{aligned} w_s^\uparrow(F) &\equiv \sup \{w(F, G) : G \in \mathcal{P}_{s,2}(M_s)\} \\ &= \sup \{w(F, G) : G \in \mathcal{P}_{s,2,3}(M_s)\} = w(F, G^*(F)). \end{aligned}$$

(c) There exists (F^{**}, G^{**}) in $\mathcal{P}_{a,2,3}(M_a) \times \mathcal{P}_{s,2,3}(M_s)$ such that

$$\begin{aligned} w^\uparrow &\equiv \sup \{w(F, G) : F \in \mathcal{P}_{a,2}, G \in \mathcal{P}_{s,2}(M_s)\} \\ &= \sup \{w(F, G) : F \in \mathcal{P}_{a,2,3}(M_a), G \in \mathcal{P}_{s,2,3}(M_s)\} \\ &= w(F^{**}, G^{**}) = w_a^\uparrow(G^{**}) = w_s^\uparrow(F^{**}). \end{aligned}$$



Ideas of Proof for Case (a)

- Start with $W \stackrel{d}{=} [W + V_G - U_F]^+$.

► $E[W] = E[(W + V_G - U_F)^+] = \int_0^{M_a} \phi(u) dF$ for ϕ expressed as the density of $W + V_G - U_F$.

$$\phi(u) \equiv \int_0^{M_a} \int_0^{M_a} (x + v - u)^+ dG(x) dF(v), \quad 0 \leq u \leq M_a.$$

- Optimize $E[W] = \int_0^{M_a} \phi(u) dF$ over $F \in \mathcal{P}_{a,2}(M_a)$. It can be written as

$$\sup \{ E[(W + V_G - U_F)^+] : F_U \in \mathcal{P}_{a,2}(M_a) \}$$

- Let $W_1 = (U_1 + V_1 - U_{F_1})^+$, $E[W_1] \geq E[U_1]$
- Also, $W_2 = (W_1 + V_2 - U_2)^+$, $E[W_2] \geq E[U_2]$



Ideas of Proof for Case (a)

- ▶ Start with $W \stackrel{d}{=} [W + V_G - U_F]^+$.
- ▶ $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for ϕ expressed as the double integral

$$\phi(u) \equiv \int_0^\infty \int_0^\infty (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.$$

- ▶ Optimize $E[W] = \int_0^{M_a} \phi(u) dF$ can be written as

$$\sup \{ E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2}(M_a) \}$$

- ▶ Optimal $F_1 = (W_1 + V - U_F)^+$, $E[W_1] \geq E[W]$
- ▶ Optimal $F_2 = (W_1 + V - U)^+$, $E[W_2] \geq E[W]$



Ideas of Proof for Case (a)

- ▶ Start with $W \stackrel{d}{=} [W + V_G - U_F]^+$.
- ▶ $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for ϕ expressed as the double integral

$$\phi(u) \equiv \int_0^\infty \int_0^\infty (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.$$

- ▶ Optimize over F : $\sup_{F \in \mathcal{P}_{a,2}(M_a)} \int_0^{M_a} \phi(u) dF$. It can be written as

$$\sup \{E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2}(M_a)\}.$$

• Optimize $F_1 = (W_1 + V_1 - U_1)^+, E[W_1] \geq E[W]$

• Repeat to obtain $F_2 = (W_1 + V_2 - U_2)^+, E[W_2] \geq E[W_1]$?



Ideas of Proof for Case (a)

- ▶ Start with $W \stackrel{d}{=} [W + V_G - U_F]^+$.
- ▶ $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for ϕ expressed as the double integral

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- ▶ Optimize over F : $\sup_{F \in \mathcal{P}_{a,2}(M_a)} \int_0^{M_a} \phi(u) dF$. It can be written as

$$\sup \{E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2}(M_a)\}.$$

- ▶ Update $W_1 = (W + V - U_{F_1})^+$. $\mathbb{E}[W_1] \geq \mathbb{E}[W]$

• Report to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $\mathbb{E}[W_2] \geq \mathbb{E}[W_1]$?



Ideas of Proof for Case (a)

- ▶ Start with $W \stackrel{d}{=} [W + V_G - U_F]^+$.
- ▶ $E[W] = E[(W + V - U)^+] = \int_0^{M_a} \phi(u) dF$ for ϕ expressed as the double integral

$$\phi(u) \equiv \int_0^\infty \int_0^\infty (x + v - u)^+ dG(v) dH(x), \quad 0 \leq u \leq M_a.$$

- ▶ Optimize over F : $\sup_{F \in \mathcal{P}_{a,2}(M_a)} \int_0^{M_a} \phi(u) dF$. It can be written as

$$\sup \{E[(W + V - U)^+] : F_U \in \mathcal{P}_{a,2}(M_a)\}.$$

- ▶ Update $W_1 = (W + V - U_{F_1})^+$. $\mathbb{E}[W_1] \geq \mathbb{E}[W]$
- ▶ Repeat to obtain $W_2 = (W_1 + V - U_{F_2})^+$. $\mathbb{E}[W_2] \geq \mathbb{E}[W_1]$?



Ideas of Proof for Case (a)

- Suppose **fixed point** exists:

$$(W(F^*, G) + V - U_{F^*})^+ \stackrel{d}{=} W(F^*, G).$$

With the property

$$\begin{aligned} & E[(W(F^*, G) + V - U_{F^*})^+] \\ &= \sup \{ E[(W(F, G) + V - U_F)^+] : F \in \mathcal{P}_{a,2}(M_a) \}. \end{aligned}$$

Since $F^* \in \mathcal{P}_{a,2}(M_a)$



Ideas of Proof for Case (a)

- Suppose **fixed point** exists:

$$(W(F^*, G) + V - U_{F^*})^+ \stackrel{d}{=} W(F^*, G).$$

With the property

$$\begin{aligned} & E[(W(F^*, G) + V - U_{F^*})^+] \\ &= \sup \{ E[(W(F, G) + V - U_F)^+] : F \in \mathcal{P}_{a,2}(M_a) \}. \end{aligned}$$

- Show $F^* \in \mathcal{P}_{a,2,3}(M_a)$.



Relate to Transient Mean Waiting Time

Theorem

(reduction to the transient mean) Consider $GI/GI/1$ queues given first two moments and bounded support,

(a) For any specified $G \in \mathcal{P}_{s,2}$, if there exists $F_n \in \mathcal{P}_{a,2,3}(M_a)$ such that

$$w_n(F_n, G) = w_{a,n}^\uparrow(G) \equiv \sup \{w_n(F, G) : F \in \mathcal{P}_{a,2}(M_a)\},$$

*then the sequence $\{F_n : n \geq 1\}$ is **tight**, so that there exists a convergent subsequence. Moreover, if F is the limit of any convergent subsequence, then F is in $\mathcal{P}_{a,2,3}(M_a)$ and F is optimal for $E[W(F, G)]$.*



Relate to Transient Mean Waiting Time

Theorem

(continued) (b) For any specified $F \in \mathcal{P}_{a,2}$, if there exists $G_n \in \mathcal{P}_{s,2}(M_a)$ such that

$$w_n(F, G_n) = w_{s,n}^\uparrow(F) \equiv \sup \{w_n(F, G) : G \in \mathcal{P}_{s,2}(M_a)\},$$

then the sequence $\{G_n : n \geq 1\}$ is tight, so that there exists a convergent subsequence. Moreover, if G is the limit of any convergent subsequence, then G is in $\mathcal{P}_{s,2,3}(M_s)$ and G is optimal for $E[W(F, G)]$.

(c) If there exists (F_n, G_n) in $\mathcal{P}_{a,2,3}(M_a) \times \mathcal{P}_{s,2,3}(M_s)$ such that

$$w_n(F_n, G_n) = w_n^\uparrow \equiv \sup \{w_n(F, G) : F \in \mathcal{P}_{a,2}(M_a), G \in \mathcal{P}_{s,2}(M_s)\},$$

then the sequence $\{(F_n, G_n) : n \geq 1\}$ is tight.



The Multinomial Representation

In the bounded support three-point distribution space,

$$P_k(\mathbf{v}, \mathbf{p}) \equiv \frac{k! p_1^{k_1} p_2^{k_2} p_3^{k_3}}{k_1! k_2! k_3!},$$

$$Q_w(\mathbf{u}, \mathbf{q}) \equiv \frac{w! q_1^{w_1} q_2^{w_2} q_3^{w_3}}{w_1! w_2! w_3!}.$$

$$E[W_j] = \frac{n-1}{n} \sum_{(\mathbf{k}, \mathbf{w}) \in \mathcal{L}} \max\{0, \sum_{i=1}^3 (k_i v_i - w_i q_i)\} P_{\mathbf{k}}(\mathbf{v}, \mathbf{p}) Q_{\mathbf{w}}(\mathbf{q}).$$



The Multinomial Representation

In the bounded support three-point distribution space,

$$P_k(\mathbf{v}, \mathbf{p}) \equiv \frac{k! p_1^{k_1} p_2^{k_2} p_3^{k_3}}{k_1! k_2! k_3!},$$

$$Q_w(\mathbf{u}, \mathbf{q}) \equiv \frac{w! q_1^{w_1} q_2^{w_2} q_3^{w_3}}{w_1! w_2! w_3!}.$$

$$E[W_n] = \sum_{k=1}^n \frac{1}{k} \sum_{(\mathbf{k}, \mathbf{w}) \in \mathcal{I}} \max \{0, \sum_{i=1}^3 (k_i v_i - w_j u_j)\} P_k(\mathbf{v}, \mathbf{p}) Q_w(\mathbf{u}, \mathbf{q}).$$



The Multinomial Representation

A **tractable** formulation for optimizing $\mathbb{E}[W_n]$:

$$\begin{aligned}
 & \text{maximize} \quad \sum_{k=1}^n \frac{1}{k} \sum_{\sum k_i=k, \sum w_j=k} \max\left(\sum_i k_i v_i - \sum_j w_j u_j, 0\right) P(k_1, k_2, k_3) Q(w_1, w_2, w_3) \\
 & \text{subject to} \quad \sum_{j=1}^3 v_j p_j = s_1, \quad \sum_{j=1}^3 v_j^2 p_j = (1 + c_s^2) s_2^2, \\
 & \quad \sum_{j=1}^3 u_j q_j = m_1, \quad \sum_{j=1}^3 u_j^2 q_j = (1 + c_a^2) m_1^2, \\
 & \quad \sum_{j=1}^3 p_j = \sum_{k=1}^3 q_k = 1, \\
 & \quad M_s \geq v_j \geq 0, \quad M_a \geq u_j \geq 0, \quad p_j \geq 0, \quad q_j \geq 0, \quad 1 \leq j \leq 3.
 \end{aligned} \tag{1}$$



The Multinomial Representation

- ▶ **all local optima** in $\mathcal{P}_{a,2,2} \times \mathcal{P}_{s,2,2}$.
- ▶ $E[W(F_0, G_{u,n})]$ is **larger** than for other local optima.
- ▶ $G_{u,n}$ denote there is a optimal $b_s^*(n, M_s)$ (not necessary to be equal to M_s).

Table: Numerical values of $E[W(F_0, G_{u,n})]$ for the queue optimization and numerical values of $b_s^*(n, M_s) = \arg \max_{b_s} E[W(F_0, G_{u,n})]$.

n	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
1	0.080	0.125	0.170	0.215	0.260	0.305
5	0.250	0.375	0.500	0.625	0.750	0.875
10	0.333	0.500	0.667	0.833	1.000	1.167
15	0.389	0.577	0.759	0.938	1.110	1.289
20	0.435	0.644	0.833	1.037	1.202	1.377
25	0.470	0.704	0.896	1.093	1.266	1.438
30	0.500	0.760	0.952	1.146	1.326	1.494
35	0.526	0.812	1.000	1.196	1.383	1.547
40	0.550	0.861	1.040	1.243	1.437	1.596
45	0.571	0.907	1.077	1.287	1.488	1.643
50	0.590	0.952	1.111	1.329	1.536	1.688
55	0.607	0.995	1.143	1.369	1.581	1.731
60	0.622	1.037	1.173	1.407	1.623	1.772
65	0.636	1.077	1.200	1.443	1.663	1.811
70	0.648	1.116	1.225	1.477	1.701	1.848
75	0.659	1.153	1.248	1.509	1.737	1.883
80	0.669	1.189	1.269	1.539	1.771	1.916
85	0.678	1.224	1.288	1.568	1.803	1.947
90	0.687	1.258	1.306	1.595	1.834	1.977
95	0.695	1.291	1.322	1.621	1.863	1.999
100	0.703	1.323	1.337	1.646	1.891	2.019



The Multinomial Representation

- ▶ **all local optima** in $\mathcal{P}_{a,2,2} \times \mathcal{P}_{s,2,2}$.
- ▶ $E[W(F_0, G_{u,n})]$ is **larger** than for other local optima.
- ▶ $G_{u,n}$ denote there is a optimal $b_s^*(n, M_s)$ (not necessary to be equal to M_s).

Table: Numerical values of $E[W_n(F_0, G_{u,n})]$ from the optimization and numerical search for $c_a^2 = c_s^2 = 4.0$ for $M_a = M_s = 10$

n	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
1	0.080	0.160	0.240	0.320	0.400	0.489	0.579	0.668	0.758
5	0.269	0.538	0.813	1.095	1.414	1.777	2.140	2.505	2.882
10	0.357	0.716	1.102	1.525	2.056	2.634	3.228	3.869	4.555
15	0.386	0.778	1.220	1.744	2.410	3.137	3.949	4.832	5.776
20	0.395	0.804	1.281	1.871	2.626	3.508	4.499	5.602	6.808
25	0.399	0.814	1.313	1.948	2.781	3.782	4.933	6.242	7.693
30	0.400	0.820	1.332	1.999	2.896	3.992	5.291	6.794	8.508
35	0.400	0.822	1.343	2.032	2.979	4.163	5.590	7.270	9.185
40	0.400	0.824	1.349	2.056	3.040	4.299	5.846	7.696	9.858
45	0.400	0.824	1.354	2.072	3.088	4.411	6.067	8.075	10.423
50	0.400	0.825	1.356	2.084	3.126	4.505	6.260	8.421	11.002



Numerics and Simulation Search over $\mathcal{P}_{a,2,2} \times \mathcal{P}_{s,2,2}$

Table: Numerical estimates of $E[W_{20}]$ as a function of b_a and b_s when $\rho = 0.5$, $c_a^2 = c_s^2 = 4.0$ and $M_a = 7 < M_s = 10$.

$b_s \backslash b_a$	5.00	5.25	5.50	5.75	6.00	6.25	6.50	6.75	7.00
5.0	2.497	2.530	2.518	2.497	2.469	2.439	2.406	2.371	2.335
5.5	2.557	2.414	2.420	2.422	2.402	2.378	2.351	2.320	2.288
6.0	2.561	2.447	2.328	2.318	2.328	2.312	2.290	2.266	2.239
7.0	2.549	2.447	2.331	2.204	2.165	2.149	2.154	2.150	2.132
8.0	2.556	2.430	2.319	2.208	2.074	2.029	2.021	2.010	2.007
9.0	2.598	2.456	2.310	2.183	2.068	1.937	1.895	1.903	1.898
10.0	2.626	2.506	2.353	2.188	2.043	1.921	1.786	1.779	1.789



Numerics and Simulation Search over $\mathcal{P}_{a,2,2} \times \mathcal{P}_{s,2,2}$

Table: Simulation estimates of $E[W]$ as a function of b_a and b_s when $\rho = 0.5$, $c_a^2 = c_s^2 = 4.0$ and $M_a = 7 < M_s = 10$.

$b_s \backslash b_a$	5.00	5.25	5.50	5.75	6.00	6.25	6.50	6.75	7.0
5.0	3.110	3.134	3.117	3.083	3.040	2.997	2.950	2.910	2.863
5.5	3.179	3.026	3.019	3.009	2.975	2.938	2.901	2.860	2.823
6.0	3.191	3.065	2.932	2.907	2.905	2.876	2.844	2.809	2.767
7.0	3.181	3.067	2.942	2.797	2.748	2.720	2.713	2.691	2.670
8.0	3.195	3.056	2.934	2.810	2.664	2.611	2.591	2.564	2.553
9.0	3.239	3.092	2.931	2.792	2.663	2.525	2.472	2.467	2.449
10.0	3.282	3.142	2.986	2.812	2.640	2.507	2.367	2.350	2.349



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- Given an arrival time dist F , what is the extremal mean arrival time dist F^* leading to the UB of $GI/GI/1$?
- Given an arrival time dist F and a service time dist G , what is the extremal three point dist F^* leading to the UB of $GI/GI/1$?
- Given an arrival time dist F , what is the extremal service time dist G^* leading to the UB of $GI/GI/1$?
- Given an arrival time dist F and a service time dist G , what is the extremal three point dist F^* leading to the UB of $GI/GI/1$?
- Given an arrival time dist F and a service time dist G , what is the extremal three point dist F^* and G^* leading to overall UB of $GI/GI/1$?
- Given an arrival time dist F and a service time dist G , what is the extremal F^* and G^* leading to overall UB of $GI/GI/1$?



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any G , what is the **extremal inter-arrival time dist** F^* attaining the UB of $\mathbb{E}[W(F, G)]$?

▶ Answer: three-point dist.

- ▶ Given any G , what is the **extremal service time dist** G^* attaining the UB of $\mathbb{E}[W(F, G)]$?

▶ Answer: three-point dist.

- ▶ Answer: F^* and G^* leading to overall UB

▶ Answer: F^*, G^*



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any G , what is the **extremal inter-arrival time dist** F^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** F^* .

Given any G , what is the **extremal service time dist** G^* attaining the UB of $\mathbb{E}[W(F, G)]$?

Answer: **three point dist**.

Given any F and G , what is the **extremal** F and G^* leading to overall UB

of $\mathbb{E}[W(F, G)]$?



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any G , what is the **extremal inter-arrival time dist** F^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** F^* .
- ▶ Given any F , what is the **extremal service time dist** G^* attaining the UB of $\mathbb{E}[W(F, G)]$?

Answer: **three point dist** G^* .

What are the extremal F^* and G^* leading to overall UB of $\mathbb{E}[W(F, G)]$?

Answer: $F^* = \delta_0$, $G^* = \delta_{\mu}$.



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any G , what is the **extremal inter-arrival time dist** F^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** F^* .
- ▶ Given any F , what is the **extremal service time dist** G^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** G^* .

What are the extremal F^* and G^* leading to overall UB of $\mathbb{E}[W(F, G)]$?

Ans: F^*, G^*



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any G , what is the **extremal inter-arrival time dist** F^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** F^* .
- ▶ Given any F , what is the **extremal service time dist** G^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** G^* .
- ▶ What are the **extremal** F^* and G^* leading to **overall** UB of $\mathbb{E}[W(F, G)]$?



Classical $GI/GI/1$ Problem

We want to address a **Long Standing Open Problem** in Queueing Theory.

- ▶ Given any G , what is the **extremal inter-arrival time dist** F^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** F^* .
- ▶ Given any F , what is the **extremal service time dist** G^* attaining the UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **three point dist** G^* .
- ▶ What are the **extremal** F^* and G^* leading to **overall** UB of $\mathbb{E}[W(F, G)]$?
 - ▶ Answer: **the pair** F_0, G_u .



Impact of Inter-arrival Time

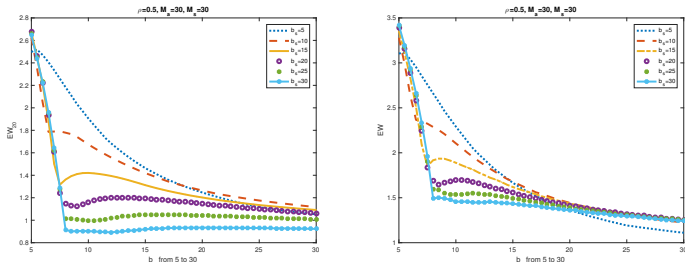


Figure: Simulation estimates of the transient mean $E[W_{20}]$ (left) and the steady-state mean $E[W]$ (right) as a function of b_a for six cases of b_s the in the case $\rho = 0.5$, $c_a^2 = c_s^2 = 4.0$ and $M_a = M_s = 30$.



Impact of Service Time

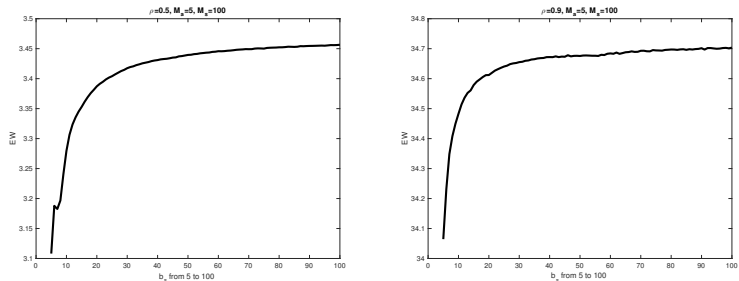


Figure: $E[W(F_0, G)]$ for $G \in \mathcal{P}_{s,2,2}$ as a function of b_s given $b_a = (1 + c_a^2)$.



Counterexamples

Conjecture

Given any $G \in \mathcal{P}_{s,2}$, the extremal inter-arrival time is F_0 .
 Given any $F \in \mathcal{P}_{a,2}$, the extremal service time is G_0, G_u .

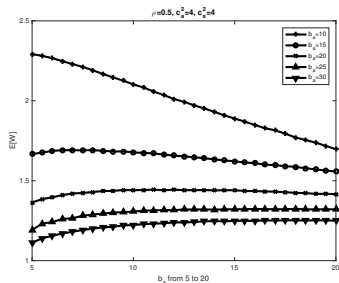
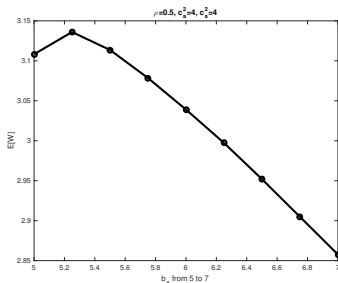


Figure: The $E[W]$ as a function of b_s in $[(1+c_s^2), 7]$ for $b_s = 5$, i.e., for F_0 (left) and as a function of b_s in $[(1+c_s^2), M_s = 20]$ for $b_s = 5$ (right).



Counterexamples

Conjecture

Given any $G \in \mathcal{P}_{s,2}$, the extremal inter-arrival time is F_0 .
 Given any $F \in \mathcal{P}_{a,2}$, the extremal service time is G_0, G_u .

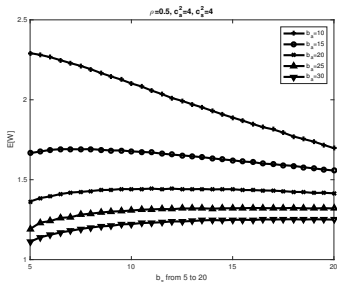
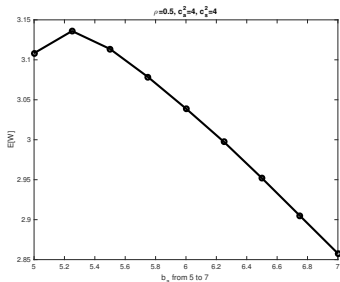


Figure: The $E[W]$: a function of b_a in $[(1 + c_a^2), M_a = 7]$ for $b_s = 5$, i.e., for G_0 (left) and as a function of b_s in $[(1 + c_s^2), M_s = 20]$ for b_a (right).



Conjectures

Theorem

(Counterexamples) Fix any service time dist G , $F^*(G) = F_0$;
Fix any inter-arrival dist F , $G^*(F)$ is G_0 or G_u . The both
arguments are *invalid*.

(Chen and Whitt I) Fix any G , the extremal $F^*(G)$ is a
two-point distribution.

(Chen and Whitt II) Fix any two-point F , the extremal $G^*(F)$
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Conjecture

(Chen and Whitt II) Fix any *two-point* F , the extremal $G^*(F)$
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Upper Bound Inequality

Overall Upper Bound:

$$\begin{aligned} E[W(F_0, G_{u^*})] &\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)} \\ &\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)}. \end{aligned}$$

$\delta \in (0, 1)$ and $\rho = \lambda/\mu = (1-\delta)/\rho$.

(an UB for $E[W(F_0, G_{u^*})]$) For the GI/GI/1 queue with parameter four-tuple $(1, c_a^2, \rho, c_s^2)$, if $E[W(F_0, G_{u^*})]$ is UB, then

$$E[W(F_0, G_{u^*})] \leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)}.$$



Upper Bound Inequality

Overall Upper Bound:

$$\begin{aligned}\mathbb{E}[W(F, G)] &\leq \mathbb{E}[W(F_0, G_{u^*})] \\ &\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)}.\end{aligned}$$

$\delta \in (0, 1)$ and $\delta = \exp(-(1-\delta)/\rho)$.

(an UB for $E[W(F_0, G_{u^})]$) For the GI/GI/1 queue with parameter four-tuple $(1, c_a^2, \rho, c_s^2)$, if $E[W(F_0, G_{u^*})]$ is UB, then*

$$E[W(F_0, G_{u^*})] \leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)}.$$



Upper Bound Inequality

Overall Upper Bound:

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Theorem

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Thank You!

