# $M^X/G/1$ queues with general vacations: decomposition results

I. Kleiner

2018

# Overview

- ❑Introduction
- ❑Queueing models with vacations
- ❑Decomposition property
- ❑Special examples

# Introduction

- Queues with vacations are important class of queues, useful to describe and analyze: computer systems, communication networks etc.

- A vacation, in queueing context, is the period when the server does not attending to a particularly targeted queue
  - The vacation starts only when the system becomes empty
  - Server can return from vacation only when at least one customer in the system

# Introduction - continue

- Since work of *Levy* and *Yechiali*: about vacation queues, researchers have studied vacations queueing systems with extensions:

  - Working (utilized) vacation policy  (Y. Levy)
  - Abandonments during vacation (U. Yechiali)
  - Synchronize reneging (I. Adan)

# Introduction – decomposition result

- For each of these examples, authors provide decomposition results for the steady states probabilities of the number of customers in the system

# Introduction – decomposition result

- Decomposition result: the steady state distribution (s.s.d.) of the number of customers in the system is a convolution of:
  - The s.s.d. of the number of customers in the corresponding system without vacation
  - The number of customers in vacation

"Understanding this decomposition is helpful to analyze some complex models"

*B.T. Doshi* **Queuing systems with vacations - a survey**

# Our Results

- Decomposition results: for the broad class of vacation systems:
  - M^X/G/1 queueing system with general vacation policies
  - M^X/M/1 queueing  system with general vacations policies

- Analysis of vacation models with balking impatience customers with rational rates

# $M^X/G/1$ - Decomposition Result

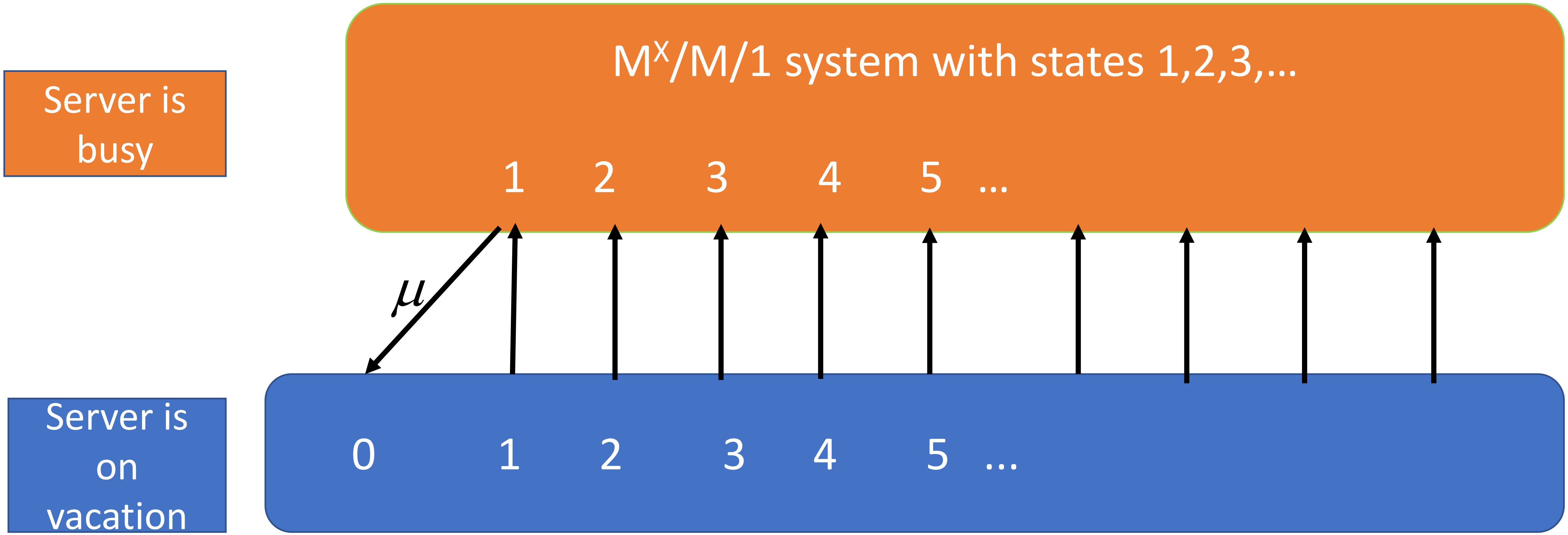
# Model Definition

- *$M^X/G/1$ queue with general vacation*
  - *Compound Poisson arrivals during working phase*
  - As soon as the system becomes empty, the server goes on vacation
  - During vacations:
    - customers may: arrive, balk, renege, leave alone or in group due to disaster etc.
    - The server can serve vacation queue in any way: in group, in one, etc.
    - At the end of vacation the customers can: leave in group, born new generation, etc.

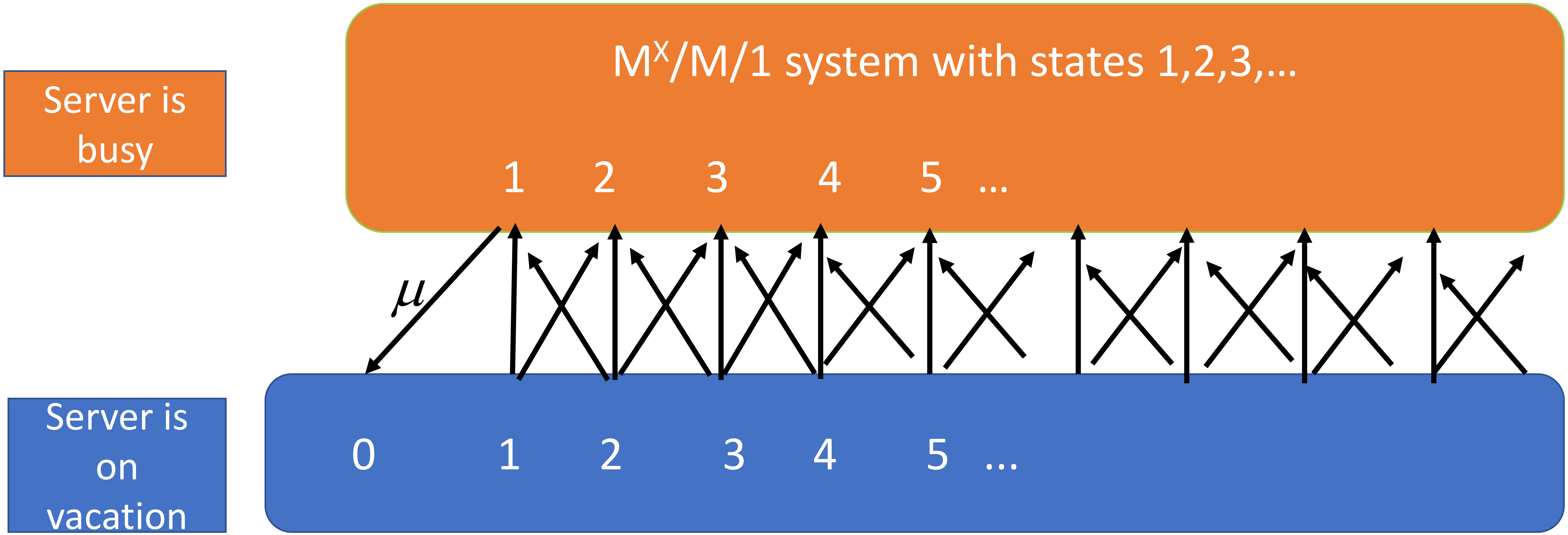
# Model Assumptions

- Existence of stationary distribution of $\Psi$ - the number of customers in the system at the end of vacation epoch
- At the end of vacation the server restarts the service of the customers

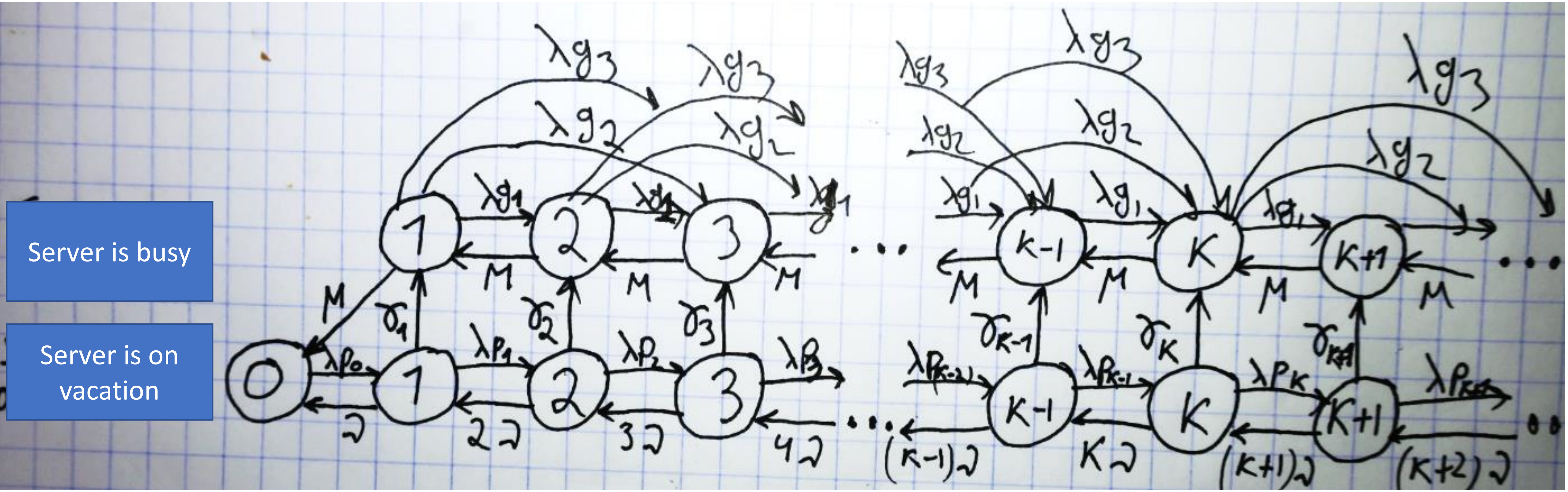
# General description: $M^X/M/1$ Model



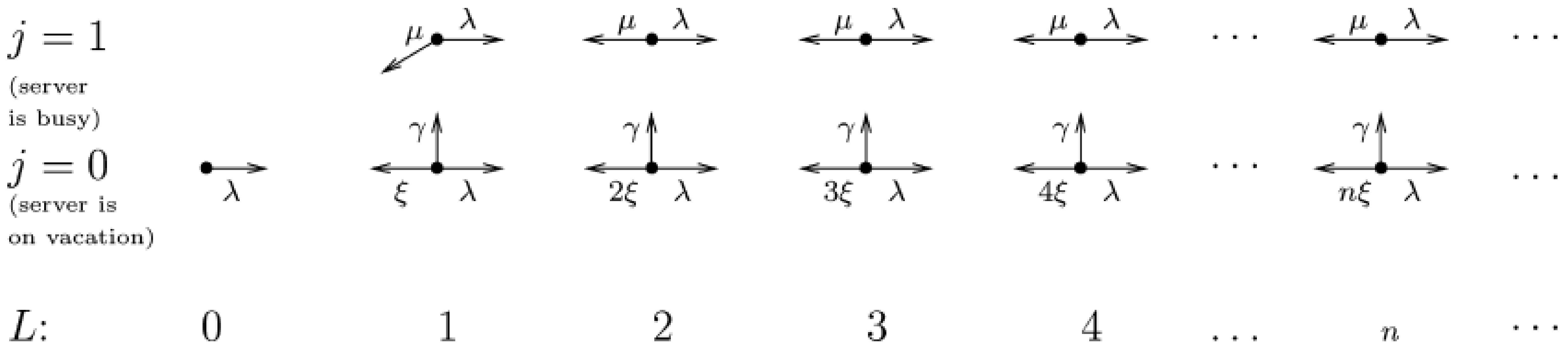
# General description: $M^X/M/1$ Model



# **Example**: $M^X/M/1$ system with impatient customers during vacation - reneging and balking



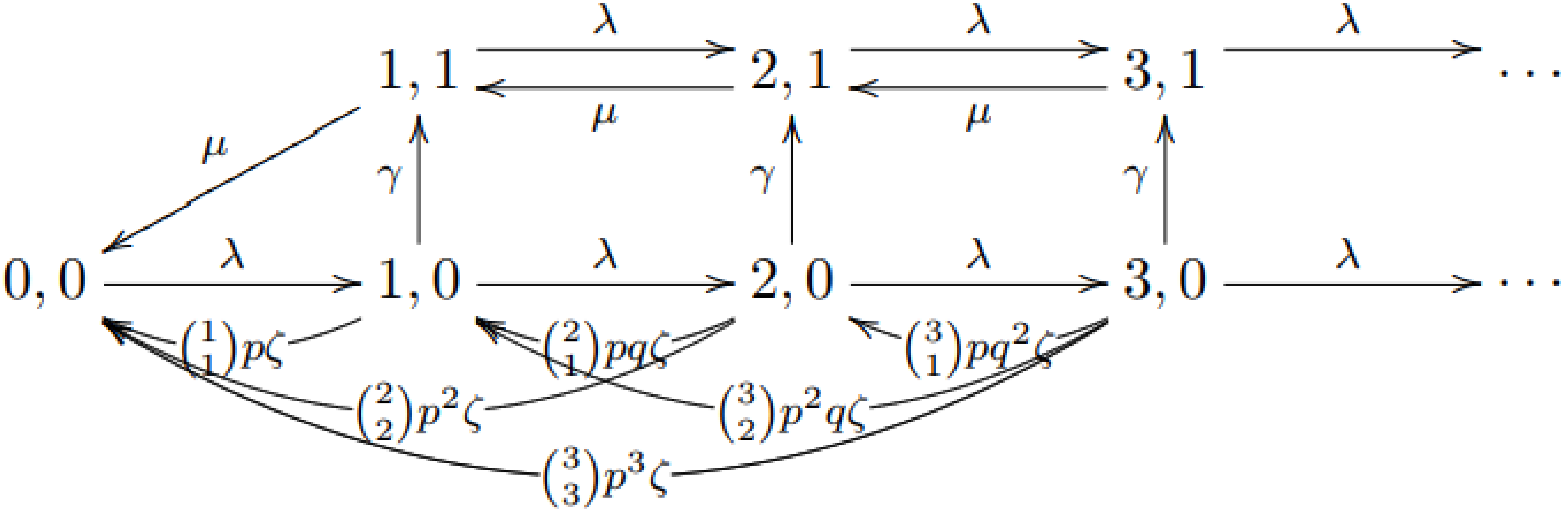
# **Example**: Impatient Customers During Vacation - reneging




**Analysis of customers' impatience in queues with server vacations**

**Eitan Altman · Uri Yechiali**

# Example: MAE model




**Synchronized reneging in queueing systems with vacations**

Ivo Adan[1], Antonis Economou[2] and Stella Kapodistria[

# Decomposition Theorem – notations:

- ***Notation***:
  - L the number of the customers in the system
  - J the phase of the system: J=0 vacation phase, J=1 working phase
  - $\pi_{n,j}$=P(L=n,J=j) - the system s.s.d. for n=j,2,3,…, j=0,1
  - $G_0(z)=\sum_{i=0}^{\infty}\pi_{n,0}z^j$ , $G_1(z)=\sum_{i=1}^{\infty}\pi_{n,1}z^j$- partial probability generating functions pgf.
  - $\Psi$ the number of customers in the system at the end of vacation epoch
  - $G_g(z)=\sum_{i=1}^{\infty}g_i z^j$- the probability generating function of the arrival group size

# Decomposition Theorem $M^X/G/1$

- ***Theorem***:

$$\frac{G_1(z)}{P(J=1)} = G^{\text{mod}}_{M^X/G/1}(z) \quad \cdot \quad \frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)}$$

g.f. of conditional numbers of customers in corresponding $M^X/G/1$ , given that server in state 1

G.F. of equilibrium forward recurrence

# Decomposition Theorem M/G/1

❑ ***Theorem***:

$$\frac{G_1(z)}{P(J=1)} = \frac{G_{M/G/1}(z)-(1-\rho)}{\rho} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

$$\frac{G_1(z)}{P(J=1)} = \underbrace{\frac{(1-\rho)}{\rho} z \frac{1-\tilde{B}(\lambda(1-z))}{\tilde{B}(\lambda(1-z))-z}}_{\text{Conditional number of customers in standard M/G/1, given that the system in working phase (Pollaczek-Khinnchin)}} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

# Decomposition Theorem $M^X/M/1$

- ***Theorem***:

$$\frac{G_1(z)}{P(J=1)} = z \cdot G_{M^X/M/1}(z) \quad \cdot \quad \frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)}$$

G.F. of mod $M^X/M/1$

G.F. of equilibrium forward recurrence

# Decomposition Theorem $M^X/M/1$

❑ ***Theorem***:

First form

$$\frac{G_1(z)}{P(J=1)} = z \cdot G_{M^X/M/1}(z) \cdot \frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)}$$

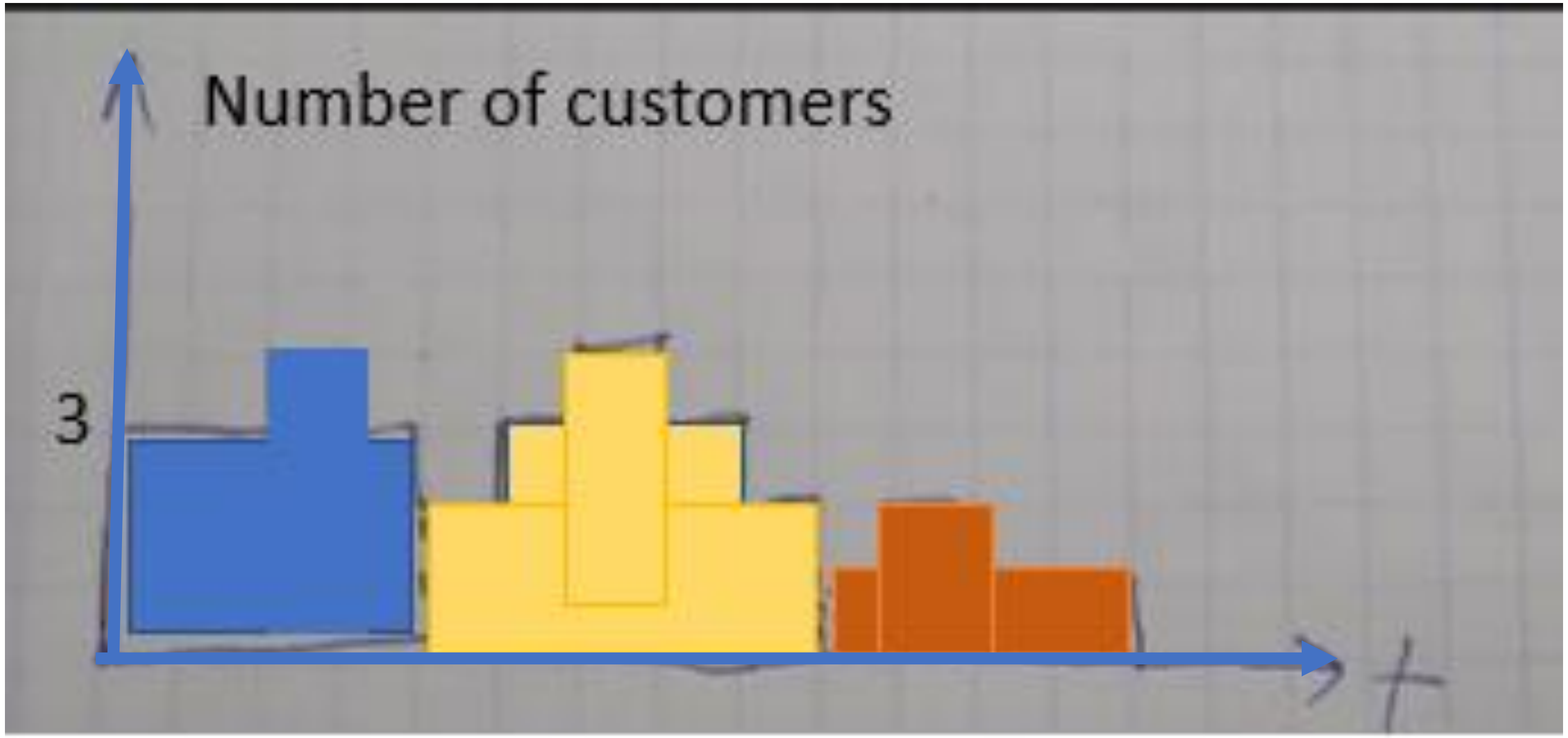
Second form

$$\frac{G_1(z)}{P(J=1)} = \frac{G_{M^X/M/1}(z)-(1-\rho)}{\rho} \cdot \frac{(1-z)G`_g(1)}{1-G_g(z)} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

$$G_{M^X/M/1}(z) = \frac{(1-\rho)}{1-\rho z \dfrac{1-G_g(z)}{(1-z)G'_g(1)}}$$

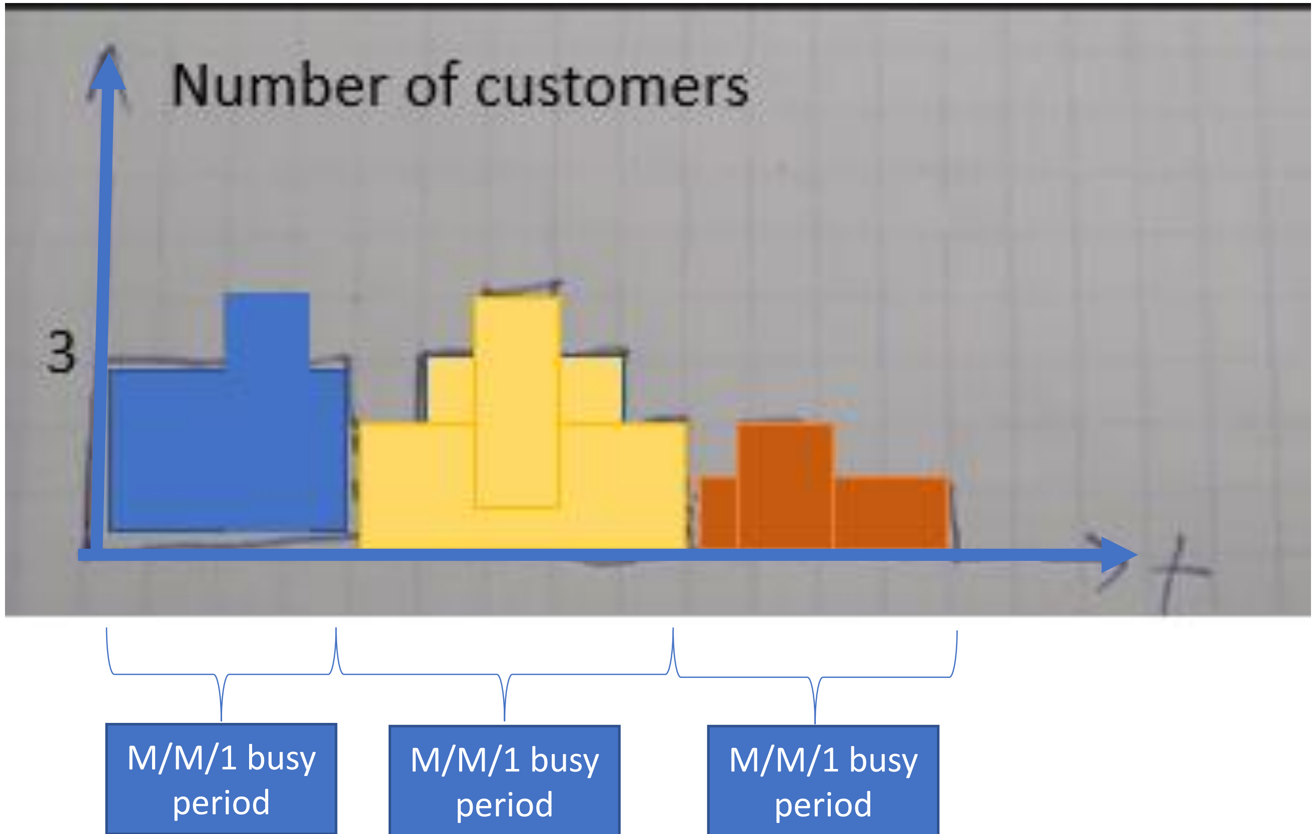
# Decomposition Theorem $M^X/G/1$

- **Proof**: Simple case – **M/M/1** system, vacations phase always ends with 3 customers in the system
- **Example:** one cycle sample realization



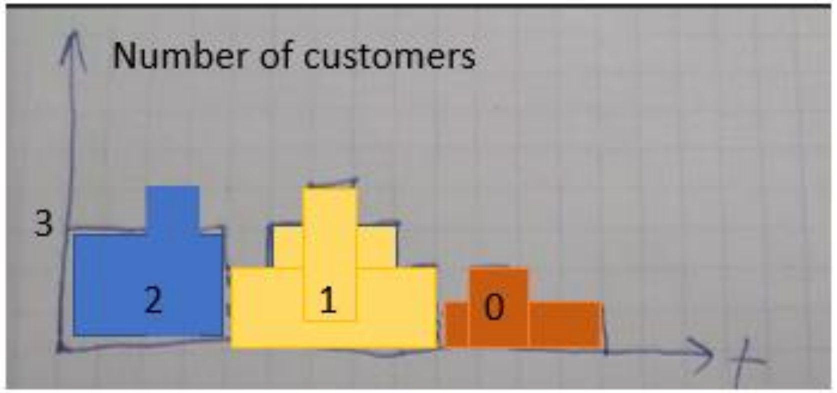
# Decomposition Theorem $M^X/G/1$

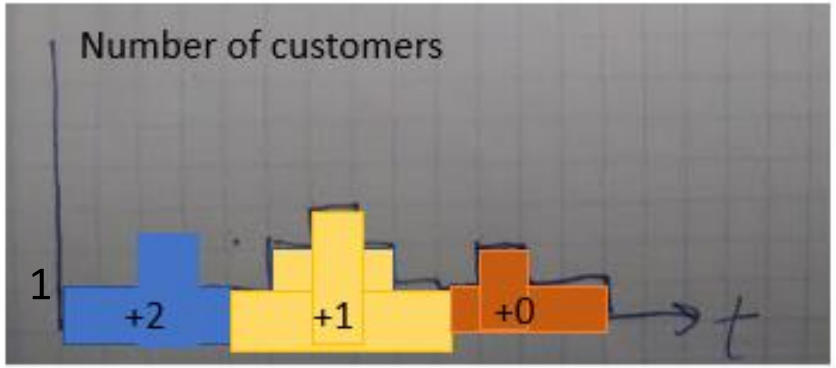
- **Proof**: Simple case – **M/M/1** system, vacations phase always ends with 3 customers in the system
- **Example:** one cycle sample realization



# Decomposition Theorem $M^X/G/1$

- **Proof**: Simple case – M/M/1 system, vacations always end with 3 customers in the system
- **Example**: one cycle sample realization

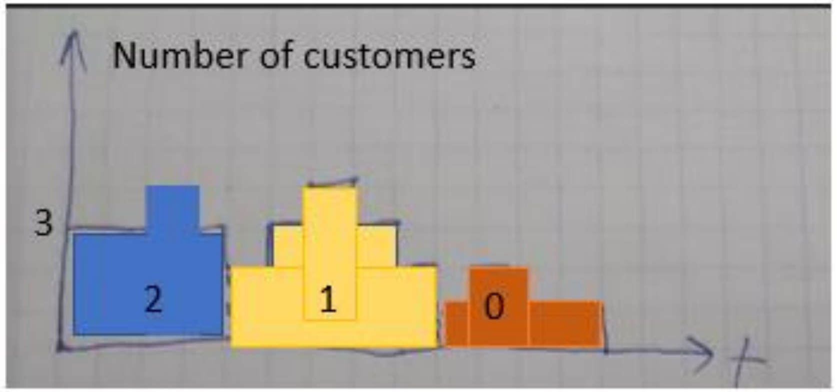


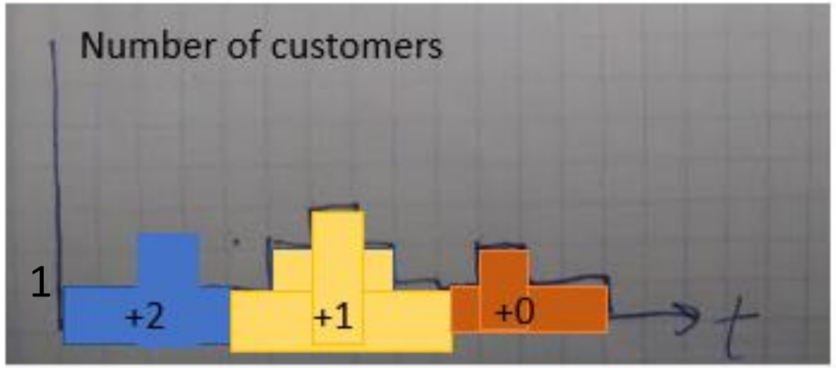




$$\frac{G_1(z)}{P(J=1)} = z\,\frac{\mu-\lambda}{\mu-\lambda z} \quad \cdot \quad \left(\frac{1}{3}+\frac{z}{3}+\frac{z^2}{3}\right)$$

# Decomposition Theorem $M^X$/G/1

- **Proof**: Simple case – M/M/1 system, vacations always end with 3 customers in the system
- **Example**: one cycle sample realization




$$\frac{G_1(z)}{P(J=1)} = z\frac{\mu-\lambda}{\mu-\lambda z} \cdot \left(\frac{1}{3}+\frac{z}{3}+\frac{z^2}{3}\right) = z\frac{\mu-\lambda}{\mu-\lambda z}\frac{1-z^3}{3(1-z)} = z\frac{1-\lambda/\mu}{1-\lambda z/\mu}\frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

# Decomposition Theorem M$^X$/G/1

- **Proof**: Simple case – **M/M/1** system, vacations phase ends with **random number** of customers
- In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

$$\frac{G_1(z)}{P(J=1)} = z\,\frac{\mu-\lambda}{\mu-\lambda z}\,\frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

# Decomposition Theorem $M^X/G/1$

- **Proof**: **$M^X/M/1$** system, vacation ends with **random number of customers**
- In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

$$\frac{G_1(z)}{P(J=1)} = \underbrace{z\frac{(1-\rho)}{1-\rho z\dfrac{1-G_g(z)}{(1-z)G'_g(1)}}}_{\text{G.F. of MX/M/1}} \cdot \underbrace{\frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)}}_{\text{G.F. of equilibrium forward recurrence}}$$

# Decomposition Theorem $M^X/G/1$

- **Proof 2**: $M^X/M/1$ system, vacation ends with random number of customers
- It is possible to prove theorem using generating function approach and balance equations

$$G_1(z) = z\frac{\mu\pi_{1,1} - G_\Psi(z)c}{\mu(1-z) - \lambda z(1-G_g(z))}$$

$$G_1(z) = \frac{(\mu\pi_{1,1} - G_\Psi(z)c)z}{\mu(1-z) - \lambda z(1-G_g(z))} = \frac{(c - G_\Psi(z)c)z}{\mu(1-z) - \lambda z(1-G_g(z))} = \frac{(1 - G_\Psi(z))cz}{\mu(1-z) - \lambda z(1-G_g(z))}$$

$$G_1(1) = \pi_{.,1} = \frac{G`_\Psi(z)c}{\mu - \lambda G`_g(1)}$$

$$\frac{G_1(z)}{\pi_{.,1}} = \frac{(1 - G_\Psi(z))cz}{\mu(1-z) - \lambda z(1-G_g(z))}\frac{\mu - \lambda G`_g(1)}{G`_\Psi(z)c} = z\frac{\mu(1-z)\left(1 - \frac{\lambda}{\mu}G`_g(1)\right)}{\mu(1-z) - \lambda z(1-G_g(z))}\frac{(1 - G_\Psi(z))c}{(1-z)G`_\Psi(z)c}$$

$$\frac{G_1(z)}{\pi_{.,1}} = z\frac{\mu(1-z)\left(1 - \frac{\lambda}{\mu}G`_g(1)\right)}{\mu(1-z) - \lambda z(1-G_g(z))}\frac{1 - G_\Psi(z)}{(1-z)G`_\Psi(z)} = z\frac{\mu(1-z)(1-\rho)}{\mu(1-z) - \lambda z(1-G_g(z))} \cdot \frac{1 - G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

# Decomposition Theorem M/G/1

- **Proof**: **M/G/1** system, vacation ends with **random number of customers**
- In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

$$\frac{G_1(z)}{P(J=1)} = \frac{G_{M/G/1}(z) - (1-\rho)}{\rho} \cdot \frac{1 - G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

# Decomposition Theorem M/G/1

- **Proof**: **M/G/1** system, vacation ends with **random number of customers**
- In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

$$\frac{G_1(z)}{P(J=1)} = \frac{G_{M/G/1}(z)-(1-\rho)}{\rho} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

$$\frac{G_1(z)}{P(J=1)} = \frac{(1-\rho)}{\rho}\, z\, \frac{1-\tilde{B}(\lambda(1-z))}{\tilde{B}(\lambda(1-z))-z} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

# Decomposition Theorem $M^X/G/1$

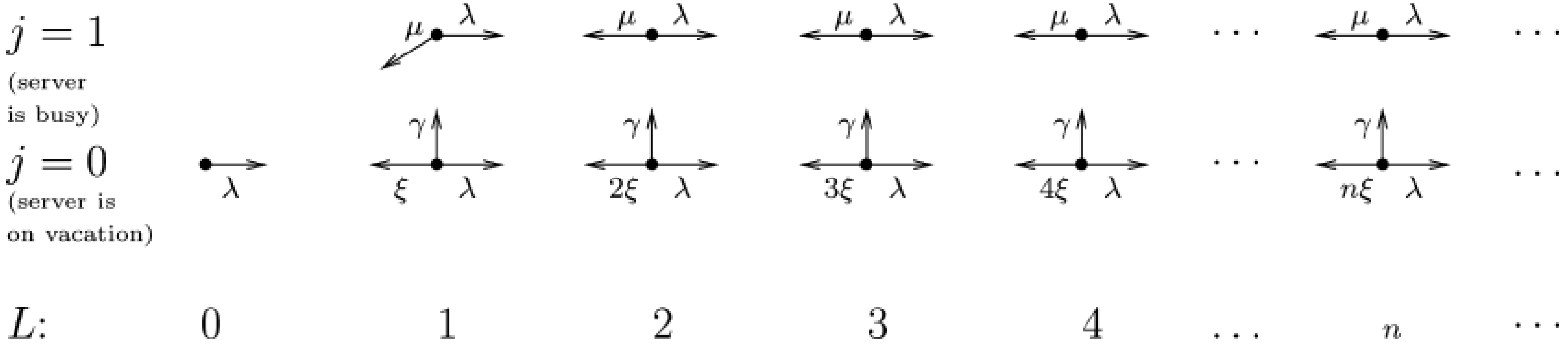
❑**Under investigation**

$$\frac{G_1(z)}{P(J=1)} = G^{\text{mod}}_{M^X/G/1}(z) \cdot \frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)}$$

g.f. of conditional numbers of customers in corresponding $M^X/G/1$, given that server in state 1

G.F. of equilibrium forward recurrence

# **Example**: M/M/1 Vacation Server with reneging impatience




$$\frac{G_1(z)}{P(J=1)} = z\,\frac{\mu-\lambda}{\mu-\lambda z}\,\frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

$$G_1(z) = \frac{\gamma G_0(z)z - Az}{(\lambda z-\mu)(1-z)}, \qquad \gamma P_{0\bullet} = A,$$

$$G_\Psi(z) = \frac{\gamma\sum_{i=1}^{\infty}\pi_{i,0}z^j}{\gamma\sum_{i=1}^{\infty}\pi_{i,0}} = \frac{G_0(z)-\pi_{00}}{\pi_{.,0}-\pi_{00}}$$

**Analysis of customers' impatience in queues with server vacations**

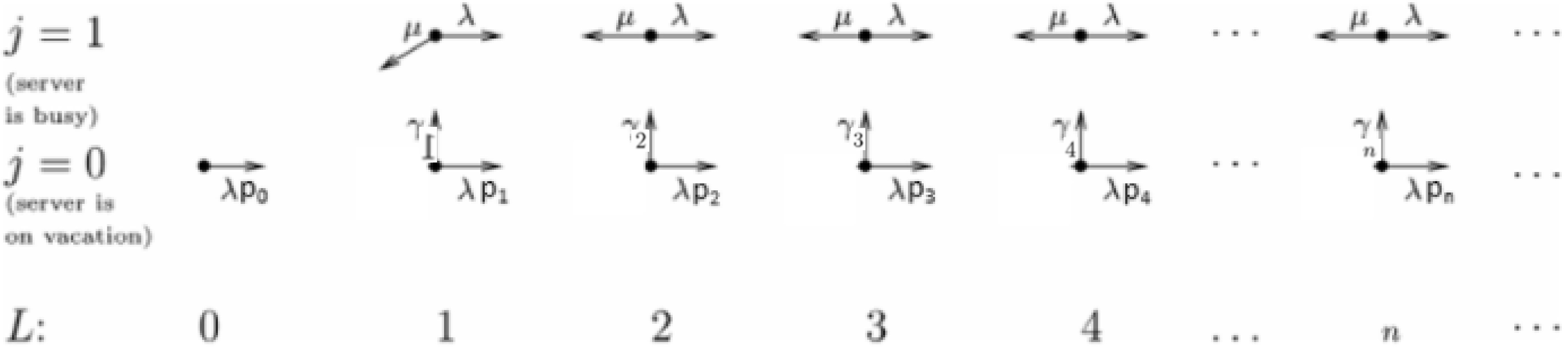
**Eitan Altman · Uri Yechiali**

# Special Example

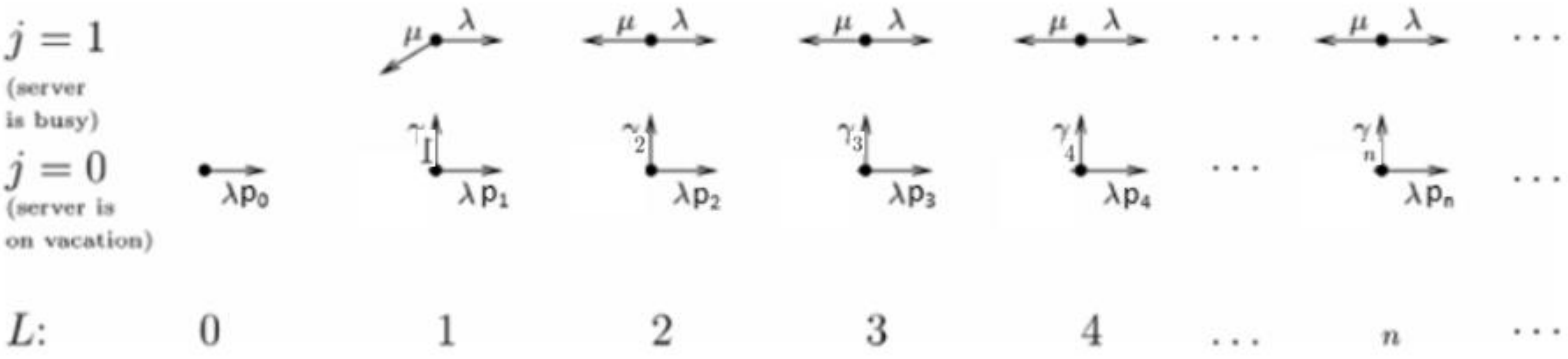
# M/M/1 vacation system with balking impatience customers and state depend vacations

- *M/M/1 server queue with multiple vacations*
  - *Poisson arrivals during working phase*
  - As soon as the system becomes empty, the server goes on multiply vacation
  - The vacation time is a random exponential variable with the density $\gamma_i$, where *i* is the number of customers in the system
  - During vacation arriving customer joins at system with probability $p_i$ when he sees *i* customers in the system

# M/M/1 Vacation Server with balking impatience customers



# M/M/1 Vacation Server with Balking Impatience Customers




$$\frac{G_1(z)}{P(J=1)} = z\,\frac{\mu-\lambda}{\mu-\lambda z} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

$$G_\Psi(z) = \frac{\sum_{i=1}^{\infty}\gamma_i\pi_{i,0}z^j}{c} = \frac{\sum_{i=1}^{\infty}\beta_i z^j}{c} \qquad \beta_i = \gamma_i\pi_{i,0}$$

# Solution

- *When $\{\gamma_i\}$ and $\{p_i\}$ are rational function of i,* $G_\psi(z)$ *can be obtained by solution of second order differential equation*
- *The formula for* $G_\psi(z)$ *also can be obtained using hypergeometric functions*
- *Denote*

$$r_k = \frac{\lambda p_i}{\lambda p_{i+1} + \gamma_{i+1}} \frac{\gamma_{i+1}}{\gamma_i}$$

# Solution

- *When {$\gamma_i$} and {$p_i$} are rational function of i,* $G_\psi(z)$ *can be obtained by solution of second order differential equation*
- *The formula for* $G_\psi(z)$ *also can be obtained using hypergeometric functions*
- *Then:*

$$r_k = \frac{\lambda p_i}{\lambda p_{i+1} + \gamma_{i+1}} \frac{\gamma_{i+1}}{\gamma_i} = \frac{\prod_{i=1}^{m}(k+a_i)}{(k+1)\prod_{i=1}^{n}(k+b_i)}$$

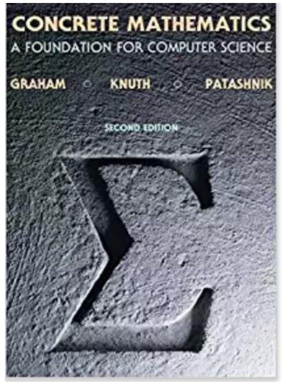
# Solution

- *When $\{\gamma_i\}$ and $\{p_i\}$ are rational function of i,* $G_\psi(z)$ *can be obtained by solution of second order differential equation*
- *The formula for* $G_\psi(z)$ *also can be obtained using hypergeometric functions*
- *Then:*

$$r_k = \frac{\lambda p_i}{\lambda p_{i+1} + \gamma_{i+1}} \frac{\gamma_{i+1}}{\gamma_i} = \frac{\prod_{i=1}^{m}(k+a_i)}{(k+1)\prod_{i=1}^{n}(k+b_i)}$$

- *In this case:*

$$G_\Psi(z) = cF\left(\begin{matrix} a_1, a_2, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix}; \lambda z\right)$$



# M/M/1 Vacation system with balking impatience customers - **demonstration**

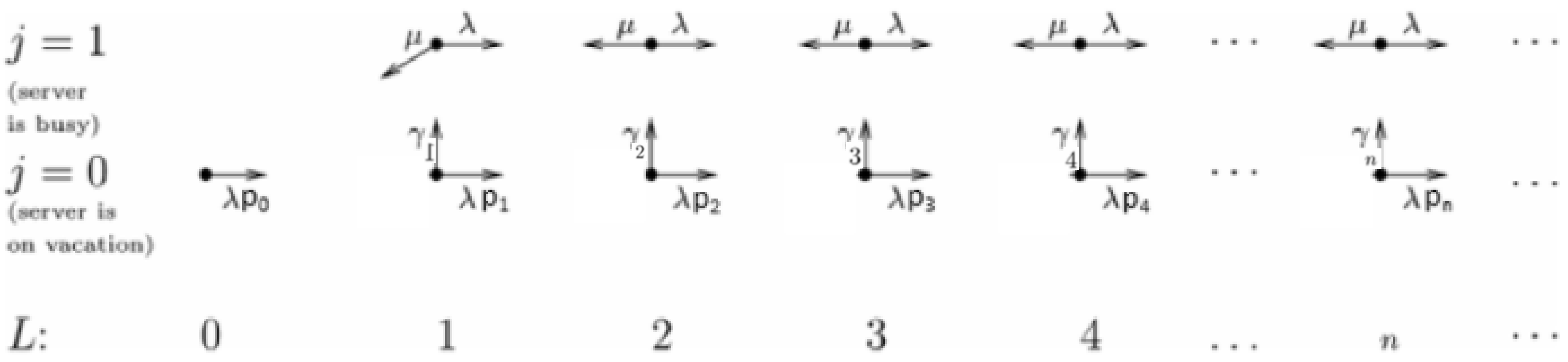
$$\gamma_i = i+1,\ p_i = \frac{1}{i+1}$$

$$\frac{\beta_{i+1,0}}{\beta_{i,0}} = \frac{\lambda(i+2)(i+2)}{(i+1)(i+1)\left(i+\frac{3}{2}+\frac{\sqrt{1-4\lambda}}{2}\right)\left(i+\frac{3}{2}-\frac{\sqrt{1-4\lambda}}{2}\right)}$$

$$G_\Psi(z) = cF\left(\begin{matrix} 2,2 \\ 1,-\frac{3}{2}+\frac{\sqrt{1-4\lambda}}{2},-\frac{3}{2}-\frac{\sqrt{1-4\lambda}}{2} \end{matrix} \mid \lambda z\right)$$

$$\frac{G_1(z)}{P(J=1)} = z\,\frac{\mu-\lambda}{\mu-\lambda z} \quad \cdot \quad \frac{1-cF\left(\begin{matrix} 2,2 \\ 1,-\frac{3}{2}+\frac{\sqrt{1-4\lambda}}{2},-\frac{3}{2}-\frac{\sqrt{1-4\lambda}}{2} \end{matrix} \mid \lambda z\right)}{(1-z)G`_\Psi(1)}$$

# M/M/1 Vacation Server with Balking Impatience Customers - **Conclusion**




For rational parameters the resulting G.F. is the product of G.F. of M/M/1 queue with corresponding hypergeometric function

# Bibliography


- *Y. Levy, U. Yechiali "***Utilization of idle time in an *M*/*G*/1 queueing system"**
- *Altman and Yechiali* **Analysis of customers impatience in queues with server vacation**
- *I. Adan A. Economou S. Kapodistria* **Synchronized reneging in queueing systems with vacations**
- *Ho Woo Lee, Soon Seok Lee, Kyung C. ChaeR,* **On a batch service queue with single vacation**
- *A. Borthakur***, On a batch arrival poison queue with generalized vacation**

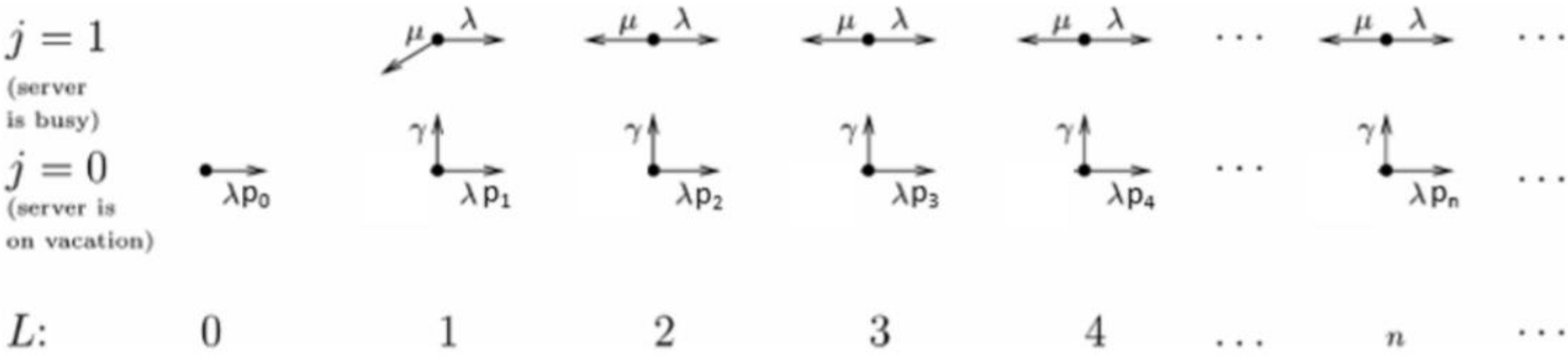
# THE END

## Primary definition (2 formulas)

$$ {}_2F_2(a_1, a_2; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \, (a_2)_k \, z^k}{(b_1)_k \, (b_2)_k \, k!} $$

# Vacation server with balking during vacation and identical $\gamma$

- For this model we found quantities of interest



# Decomposition Theorem M^X/G/1

- $G_1^+(z)$ - the g.f. of number customers leaving at departure of random customer in working phase
- Using similar approach we proved next theorem
- **Theorem 2**:

$$G^+_1(z) = \frac{1 - G_\Psi(z)}{(1-z)G'_\Psi(1)} \frac{1 - G_g(z)}{(1-z)G'_g(1)} G_1(z)$$

# Vacation server with balking during vacation and identical $\gamma$

- For this model we found quantities of interest

$$\pi_{0,0} = \left( \frac{\gamma}{\mu - \lambda} \sum_{i=1}^{\infty} \frac{i\lambda^i \prod_{j=0}^{i-1} p_j}{\prod_{j=1}^{n} p_j \lambda + \gamma} + \frac{\gamma + p_0 \lambda}{\gamma} \right)^{-1}$$

$$EL_0 = \frac{\mu - \lambda}{\gamma} \left( 1 - \frac{\gamma + p_0 \lambda}{\gamma} \pi_{0,0} \right)$$

$$EBP = \sum_{i=1}^{\infty} \frac{(\mu - \lambda + i\gamma) \prod_{j=1}^{i-1} (\lambda p_j)}{\prod_{j=1}^{i} (\gamma + \lambda p_j)}$$

$$\pi_{n,1} = \frac{\lambda}{\mu} \left[ p_{n-1} \pi_{n-1,0} + \sum_{1 \le j < n} \left( \frac{\lambda}{\mu} \right)^{n-j} p_{j-1} \pi_{j-1,0} \right]$$

# Decomposition Theorem M^X/G/1

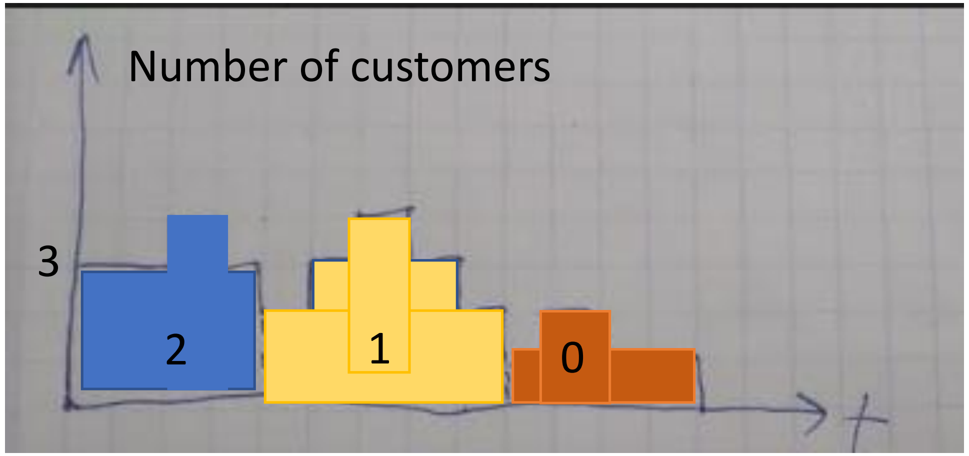
- $G_1^+(z)$ - the g.f. of number customers leaving at departure of random customer in working phase
- Using similar approach we proved next theorem
- **Theorem 2**:

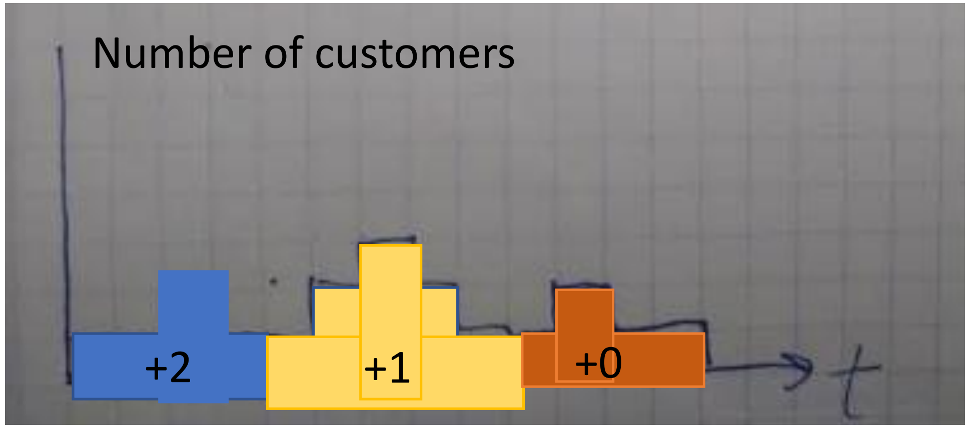
$$G^+_1(z) = \frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)} \frac{1-G_g(z)}{(1-z)G'_g(1)} G_1(z)$$

# Decomposition Theorem M^X/M/1

- ***Theorem***: $$\frac{G_1(z)}{P(J=1)} = z\frac{\mu(1-z)(1-\rho)}{\mu(1-z)-\lambda z(1-G_g(z))} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

- **Proof**: Simple case – M/M/1 system, vacations always end with 3 customers in the system



# Decomposition Theorem M^X/M/1

❑ ***Theorem***:

First form

$$\frac{G_1(z)}{P(J=1)} = z \cdot \frac{(1-\rho)}{1-\rho z \dfrac{1-G_g(z)}{(1-z)G'_g(1)}} \cdot \frac{1-G_\Psi(z)}{(1-z)G'_\Psi(1)}$$

Second form

$$\frac{G_1(z)}{P(J=1)} = \frac{G_{M^X/M/1}-(1-\rho)}{\rho} \cdot \frac{(1-z)G`_g(1)}{1-G_g(z)} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$

Third form

$$\frac{G_1(z)}{P(J=1)} = z\frac{\mu(1-z)(1-\rho)}{\mu(1-z)-\lambda z(1-G_g(z))} \cdot \frac{1-G_\Psi(z)}{(1-z)G`_\Psi(1)}$$