$M^X/G/1$ queues with general vacations: decomposition results

I. Kleiner

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Overview

- Introduction
- Queueing models with vacations
- Decomposition property
- Special examples
Introduction

- Queues with vacations are important class of queues, useful to describe and analyze: computer systems, communication networks etc.

- A vacation, in queueing context, is the period when the server does not attending to a particularly targeted queue.
  - The vacation starts only when the system becomes empty.
  - Server can return from vacation only when at least one customer in the system.
Since work of Levy and Yechiali: about vacation queues, researchers have studied vacations queueing systems with extensions:

- Working (utilized) vacation policy (Y. Levy)
- Abandonments during vacation (U. Yechiali)
- Synchronize reneging (I. Adan)
Introduction – decomposition result

For each of these examples, authors provide decomposition results for the steady states probabilities of the number of customers in the system
Introduction – decomposition result

- **Decomposition result**: the steady state distribution (s.s.d.) of the number of customers in the system is a convolution of:
  - The s.s.d. of the number of customers in the corresponding system without vacation
  - The number of customers in vacation

“Understanding this decomposition is helpful to analyze some complex models”

_B.T. Doshi Queuing systems with vacations - a survey_
Our Results

- **Decomposition** results: for the broad class of vacation systems:
  - $M^X/G/1$ queueing system with general vacation policies
  - $M^X/M/1$ queueing system with general vacations policies

- Analysis of vacation models with balking impatience customers with rational rates
\( M^x/G/1 \) - Decomposition Result
Model Definition

- $M^X/G/1$ queue with general vacation
  - Compound Poisson arrivals during working phase
  - As soon as the system becomes empty, the server goes on vacation
  - During vacations:
    - customers may: arrive, balk, renege, leave alone or in group due to disaster etc.
    - The server can serve vacation queue in any way: in group, in one, etc.
    - At the end of vacation the customers can: leave in group, born new generation, etc.
Model Assumptions

- Existence of stationary distribution of $\Psi$ - the number of customers in the system at the end of vacation epoch

- At the end of vacation the server restarts the service of the customers
General description: $M^X/M/1$ Model

$M^X/M/1$ system with states 1, 2, 3, ...

Server is busy

Server is on vacation

$\mu$
General description: $M^X/M/1$ Model

$M^X/M/1$ system with states 1,2,3,...

Server is busy

Server is on vacation

\[ \mu \]
Example: $M^X/M/1$ system with impatient customers during vacation - reneging and balking
Example: Impatient Customers During Vacation - reneging

\[ j = 1 \]
(server is busy)

\[ j = 0 \]
(server is on vacation)

\[ L: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n \quad \ldots \]

Analysis of customers’ impatience in queues with server vacations
Eitan Altman · Uri Yechiali
Example: MAE model

Synchronized reneging in queueing systems with vacations

Ivo Adan¹, Antonis Economou² and Stella Kapodistria³

¹, ², ³ — international contributors
Decomposition Theorem – notations:

- **Notation:**
  - $L$ the number of the customers in the system
  - $J$ the phase of the system: $J=0$ vacation phase, $J=1$ working phase
  - $\pi_{n,j} = P(L=n,J=j)$ - the system s.s.d. for $n=j,2,3,...$, $j=0,1$

- $G_0(z) = \sum_{i=0}^{\infty} \pi_{n,0} z^i$, $G_1(z) = \sum_{i=1}^{\infty} \pi_{n,1} z^i$ - partial probability generating functions pgf.
- $\Psi$ the number of customers in the system at the end of vacation epoch
- $G_g(z) = \sum_{i=1}^{\infty} g_i z^i$ - the probability generating function of the arrival group size
Decomposition Theorem $M^X/G/1$

**Theorem:**

\[
\frac{G_1(z)}{P(J = 1)} = G_{M^X/G/1}^{\text{mod}}(z) \cdot \frac{1 - G_{\Psi}(z)}{(1 - z)G'_{\Psi}(1)}
\]

- g.f. of conditional numbers of customers in corresponding $M^X/G/1$, given that server in state 1
- G.F. of equilibrium forward recurrence
Decomposition Theorem M/G/1

**Theorem:**

\[
\frac{G_1(z)}{P(J = 1)} = \frac{G_{M/G/1}(z) - (1 - \rho)}{\rho} \cdot \frac{1 - G_\psi(z)}{(1 - z)G^\psi_\psi(1)}
\]

\[
\frac{G_1(z)}{P(J = 1)} = \frac{(1 - \rho) - \tilde{B}(\lambda(1 - z))}{\rho} \cdot \frac{1 - \tilde{B}(\lambda(1 - z) - z)}{\tilde{B}(\lambda(1 - z)) - z} \cdot \frac{1 - G_\psi(z)}{(1 - z)G^\psi_\psi(1)}
\]

Conditional number of customers in standard M/G/1, given that the system is in working phase (Pollaczek-Khinnchin)
Decomposition Theorem $M^X/M/1$

**Theorem:**

\[
\frac{G_1(z)}{P(J = 1)} = z \cdot G_{M^x/M/1}(z) \cdot \frac{1 - G_\Psi(z)}{(1 - z)G'_\Psi(1)}
\]

G.F. of mod $M^X/M/1$

G.F. of equilibrium forward recurrence
Decomposition Theorem $M^X/M/1$

**Theorem:**

\[
\frac{G_1(z)}{P(J = 1)} = z \cdot G_{M^X/M/1}(z) \cdot \frac{1 - G_\Psi(z)}{(1 - z)G'_\Psi(1)}
\]

**First form**

\[
\frac{G_1(z)}{P(J = 1)} = \frac{G_{M^X/M/1}(z) - (1 - \rho)}{\rho} \cdot \frac{(1 - z)G'_{g}(1)}{1 - G_{g}(z)} \cdot \frac{1 - G_\Psi(z)}{(1 - z)G'_\Psi(1)}
\]

\[
G_{M^X/M/1}(z) = \frac{(1 - \rho)}{1 - \rho z} \cdot \frac{1 - G_{g}(z)}{(1 - z)G'_{g}(1)}
\]

**Second form**

The text contains mathematical formulas related to queueing theory, specifically the Decomposition Theorem for a $M^X/M/1$ queueing system. The theorem is presented in two forms, with the first form providing a general expression for the probability of a customer being served in one step ($P(J = 1)$) and the second form offering a detailed breakdown into simpler components. The expressions involve generating functions and probabilities, typical in the analysis of queueing models.
Decomposition Theorem $M^X/G/1$

- **Proof:** Simple case – $M/M/1$ system, vacations phase always ends with 3 customers in the system.
- **Example:** one cycle sample realization.
Decomposition Theorem $M^X/G/1$

**Proof:** Simple case – $M/M/1$ system, vacations phase always ends with 3 customers in the system

**Example:** one cycle sample realization
Decomposition Theorem $M^X/G/1$

**Proof:** Simple case – M/M/1 system, vacations always end with 3 customers in the system

**Example:** one cycle sample realization

\[
\frac{G_1(z)}{P(J = 1)} = z \frac{\mu - \lambda}{\mu - \lambda z} \cdot \left( \frac{1}{3} + \frac{z}{3} + \frac{z^2}{3} \right)
\]
Decomposition Theorem $M^X/G/1$

- **Proof**: Simple case – $M/M/1$ system, vacations always end with 3 customers in the system

- **Example**: one cycle sample realization

\[
\frac{G_1(z)}{P(J = 1)} = z \frac{\mu - \lambda}{\mu - \lambda z} \cdot \left( \frac{1}{3} + \frac{z}{3} + \frac{z^2}{3} \right) = z \frac{\mu - \lambda}{\mu - \lambda z} \frac{1 - z^3}{3(1 - z)} = z \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{\lambda z}{\mu}} \frac{1 - G_\Psi(z)}{(1 - z)G_\Psi(1)}
\]
Decomposition Theorem $M^X/G/1$

**Proof:** Simple case – $M/M/1$ system, vacations phase ends with random number of customers

In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

\[
\frac{G_1(z)}{P(J = 1)} = z \frac{\mu - \lambda}{\mu - \lambda z} \frac{1 - G_\psi(z)}{(1 - z)G_\psi(1)}
\]
Decomposition Theorem $M^X/G/1$

- **Proof**: $M^X/M/1$ system, vacation ends with **random number of customers**
- In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

\[
\frac{G_1(z)}{P(J = 1)} = \frac{(1 - \rho)}{1 - \rho z} \frac{1 - G_g(z)}{(1 - z)G'_g(1)} . \frac{1 - G_\Psi(z)}{(1 - z)G'_\Psi(1)}
\]

- G.F. of $MX/M/1$
- G.F. of equilibrium forward recurrence
Decomposition Theorem $M^X/G/1$

**Proof 2:** $M^X/M/1$ system, vacation ends with random number of customers

It is possible to prove theorem using generating function approach and balance equations

\[
G_1(z) = z \frac{\mu \pi_{1,1} - G_\psi(z)c}{\mu(1-z) - \lambda z(1-G_g(z))}
\]

\[
G_1(z) = \frac{(\mu \pi_{1,1} - G_\psi(z)c)z}{\mu(1-z) - \lambda z(1-G_g(z))} = \frac{(c - G_\psi(z)c)z}{\mu(1-z) - \lambda z(1-G_g(z))} = \frac{(1-G_\psi(z))cz}{\mu(1-z) - \lambda z(1-G_g(z))}
\]

\[
G_1(1) = \pi_{1,1} = \frac{G_\psi(z)c}{\mu - \lambda G_g(1)}
\]

\[
\frac{G_1(z)}{\pi_{1,1}} = z \frac{\mu(1-z) - \lambda z(1-G_g(z))}{\mu - \lambda G_g(1)} = z \frac{\mu(1-z) \left(1 - \frac{\lambda}{\mu} G_g(1)\right)}{1 - \lambda z(1-G_g(z)) G_\psi(z)} c
\]

\[
\frac{G_1(z)}{\pi_{1,1}} = z \frac{\mu(1-z) \left(1 - \frac{\lambda}{\mu} G_g(1)\right)}{\mu(1-z) - \lambda z(1-G_g(z)) G_\psi(z)} = z \frac{\mu(1-z)(1-\rho)}{\mu(1-z) - \lambda z(1-G_g(z))} \cdot \frac{1-G_\psi(z)}{(1-z)G_\psi(1)}
\]
Decomposition Theorem M/G/1

**Proof:** M/G/1 system, vacation ends with random number of customers

In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

\[
\frac{G_1(z)}{P(J = 1)} = \frac{G_{M/G/1}(z) - (1 - \rho)}{\rho} . \frac{1 - G_\Psi(z)}{(1 - z)G_\Psi'(1)}
\]
Decomposition Theorem M/G/1

Proof: M/G/1 system, vacation ends with random number of customers

In this situation, we calculate $G_1(z)$ by conditioning on $\Psi$ and taking in the account the inspection paradox (sampling-bias correction)

\[
\frac{G_1(z)}{P(J = 1)} = \frac{G_{M/G/1}(z) - (1 - \rho)}{\rho} \cdot \frac{1 - G_\Psi(z)}{(1 - z)G_\Psi(1)}
\]

\[
\frac{G_1(z)}{P(J = 1)} = \frac{(1 - \rho)}{\rho} \cdot \frac{1 - \tilde{B}(\lambda(1 - z))}{\tilde{B}(\lambda(1 - z)) - z} \cdot \frac{1 - G_\Psi(z)}{(1 - z)G_\Psi(1)}
\]
Decomposition Theorem $M^X/G/1$

Under investigation

$$\frac{G_1(z)}{P(J = 1)} = G_{M^X/G/1}^{\text{mod}}(z) \cdot \frac{1 - G_{\Psi}(z)}{(1-z)G'_{\Psi}(1)}$$

- g.f. of conditional numbers of customers in corresponding $M^X/G/1$, given that server in state 1
- G.F. of equilibrium forward recurrence
Example: M/M/1 Vacation Server with reneging impatience

\[ j = 1 \]  
(server is busy)

\[ j = 0 \]  
(server is on vacation)

\[ L: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad n \quad \ldots \]

\[ \frac{G_1(z)}{P(J = 1)} = \frac{z}{\mu - \lambda} \frac{1 - G_\psi(z)}{\mu - \lambda z (1 - z)G_\psi(1)} \]

\[ G_\psi(z) = \frac{G_0(z) - \pi_{00}}{\gamma \sum_{i=1}^{\infty} \pi_{i,0} z^i} \]

\[ G_1(z) = \frac{\gamma G_0(z) z - Az}{(\lambda z - \mu)(1 - z)} \], \quad \gamma P_0 = A, \]

Analysis of customers’ impatience in queues with server vacations

Eitan Altman · Uri Yechiali
Special Example
M/M/1 vacation system with balking impatience customers and state depend vacations

- M/M/1 server queue with multiple vacations
  - Poisson arrivals during working phase
  - As soon as the system becomes empty, the server goes on multiply vacation
  - The vacation time is a random exponential variable with the density $\gamma_i$, where $i$ is the number of customers in the system
  - During vacation arriving customer joins at system with probability $p_i$ when he sees $i$ customers in the system
M/M/1 Vacation Server with balking impatience customers
M/M/1 Vacation Server with Balking Impatience Customers

\[
G_1(z) = \frac{1}{P(J = 1)} = \frac{\mu - \lambda}{\mu - \lambda z} \cdot \frac{1 - G_\psi(z)}{(1 - z)G_\psi(1)}
\]

\[
G_\psi(z) = \frac{\sum_{i=1}^{\infty} \gamma_i \pi_{i,0} z^j}{c} = \frac{\sum_{i=1}^{\infty} \beta_i z^j}{c} \quad \beta_i = \gamma_i \pi_{i,0}
\]
Solution

- When \( \{\gamma_i\} \) and \( \{p_i\} \) are rational function of \( i \), \( G_\psi(z) \) can be obtained by solution of second order differential equation.

- The formula for \( G_\psi(z) \) also can be obtained using hypergeometric functions.

- Denote

\[
r_k = \frac{\lambda p_i}{\lambda p_{i+1} + \gamma_{i+1}} \cdot \frac{\gamma_{i+1}}{\gamma_i}
\]
Solution

- When \( \{\gamma_i\} \) and \( \{p_i\} \) are rational function of \( i \), \( G_{\psi}(z) \) can be obtained by solution of second order differential equation.

- The formula for \( G_{\psi}(z) \) also can be obtained using hypergeometric functions.

- Then:

\[
 r_k = \frac{\lambda p_i}{\lambda p_{i+1} + \gamma_{i+1}} \frac{\gamma_{i+1}}{\gamma_i} = \frac{\prod_{i=1}^{m} (k + a_i)}{\prod_{i=1}^{n} (k + b_i)}
\]
Solution

When \( \{ \gamma_i \} \) and \( \{ p_i \} \) are rational function of \( i \), \( G_\psi(z) \) can be obtained by solution of second order differential equation.

The formula for \( G_\psi(z) \) also can be obtained using hypergeometric functions.

Then:

\[
r_k = \frac{\lambda p_i}{\lambda p_{i+1} + \gamma_{i+1}} \frac{\gamma_{i+1}}{\gamma_i} = \frac{\prod_{i=1}^{m} (k + a_i)}{(k + 1) \prod_{i=1}^{n} (k + b_i)}
\]

In this case:

\[
G_\psi(z) = cF \left( \begin{array}{c} a_1, a_2, ..., a_m \\ b_1, ..., b_n \end{array} ; \lambda z \right)
\]
M/M/1 Vacation system with balking impatience customers - demonstration

\[ \gamma_i = i + 1, \quad p_i = \frac{1}{i+1} \]

\[ \frac{\beta_{i+1,0}}{\beta_{i,0}} = \frac{\lambda(i+2)(i+2)}{(i+1)(i+1) \left( i + \frac{3}{2} + \frac{\sqrt{1-4\lambda}}{2} \right) \left( i + \frac{3}{2} - \frac{\sqrt{1-4\lambda}}{2} \right) } \]

\[ G_\psi(z) = cF \left( 1, -\frac{3}{2} + \frac{\sqrt{1-4\lambda}}{2}, \frac{3}{2} - \frac{\sqrt{1-4\lambda}}{2} \mid \lambda z \right) \]

\[ \frac{G_1(z)}{P(J = 1)} = z \frac{\mu - \lambda}{\mu - \lambda z} \cdot \frac{1 - cF \left( 1, -\frac{3}{2} + \frac{\sqrt{1-4\lambda}}{2}, \frac{3}{2} - \frac{\sqrt{1-4\lambda}}{2} \mid \lambda z \right)}{(1-z)G_\psi(1)} \]
For rational parameters the resulting G.F. is the product of G.F. of M/M/1 queue with corresponding hypergeometric function.
Bibliography

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THE END
Primary definition (2 formulas)

\[ {}_2F_2(a_1, a_2; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k z^k}{(b_1)_k (b_2)_k k!} \]
Vacation server with balking during vacation and identical $\gamma$

- For this model we found quantities of interest
Decomposition Theorem M^X/G/1

- $G^+_1(z)$ - the g.f. of number customers leaving at departure of random customer in working phase

Using similar approach we proved next theorem

**Theorem 2:**

$$G^+_1(z) = \frac{1 - G_{\Psi}(z)}{(1 - z)G'_{\Psi}(1)} \cdot \frac{1 - G_{g}(z)}{(1 - z)G'_{g}(1)} \cdot G_1(z)$$
For this model we found quantities of interest

\[
\pi_{0,0} = \left( \frac{\gamma}{\mu - \lambda} \sum_{i=1}^{\infty} \frac{i \lambda^i \prod_{j=0}^{i-1} p_j}{\prod_{j=1}^{n} p_j \lambda + \gamma} + \frac{\gamma + p_0 \lambda}{\gamma} \right)^{-1}
\]

\[
EL_0 = \frac{\mu - \lambda}{\gamma} \left( 1 - \frac{\gamma + p_0 \lambda}{\gamma} \pi_{0,0} \right)
\]

\[
EBP = \sum_{i=1}^{\infty} \frac{(\mu - \lambda + i \gamma) \prod_{j=1}^{i-1} (\lambda p_j)}{\prod_{j=1}^{i} (\gamma + \lambda p_j)}
\]

\[
\pi_{n,1} = \frac{\lambda}{\mu} \left[ p_{n-1} \pi_{n-1,0} + \sum_{1 \leq j < n} \left( \frac{\lambda}{\mu} \right)^{n-j} p_{j-1} \pi_{j-1,0} \right]
\]
Decomposition Theorem M^X/G/1

- \( G_1^+(z) \) - the g.f. of number customers leaving at departure of random customer in working phase

Using similar approach we proved next theorem

**Theorem 2:**

\[
G_1^+(z) = \frac{1 - G_\psi(z)}{(1 - z)G'_\psi(1)} \cdot \frac{1 - G_g(z)}{(1 - z)G'_g(1)} \cdot G_1(z)
\]
Decomposition Theorem M^X/M/1

**Theorem:** \( \frac{G_1(z)}{P(J = 1)} = z \frac{\mu(1 - z)(1 - \rho)}{\mu(1 - z) - \lambda z (1 - G_g(z))} \cdot \frac{1 - G_\psi(z)}{(1 - z)G_\psi(1)} \)

**Proof:** Simple case – M/M/1 system, vacations always end with 3 customers in the system
Decomposition Theorem $M^X/M/M/1$

**Theorem:**

First form

$$\frac{G_1(z)}{P(J = 1)} = z \cdot \frac{(1 - \rho)}{1 - \rho z} \cdot \frac{1 - G_g(z)}{(1 - z)G'_g(1)} \cdot \frac{1 - G_\psi(z)}{(1 - z)G'_\psi(1)}$$

Second form

$$\frac{G_1(z)}{P(J = 1)} = \frac{G_{M^X/M/1}}{\rho} - (1 - \rho) \cdot \frac{(1 - z)G'_g(1)}{1 - G_g(z)} \cdot \frac{1 - G_\psi(z)}{(1 - z)G'_\psi(1)}$$

Third form

$$\frac{G_1(z)}{P(J = 1)} = z \cdot \frac{\mu (1 - z)(1 - \rho)}{\mu(1 - z) - \lambda z(1 - G_g(z))} \cdot \frac{1 - G_\psi(z)}{(1 - z)G'_\psi(1)}$$