# Strategic bidding in a discrete accumulating priority queue

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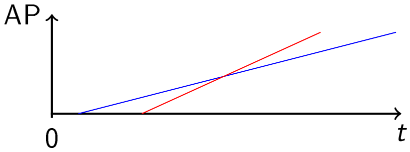
## Accumulating priority queue

- Customer accumulate priority at a constant rate while in the system
- One with the greatest (accumulated) priority is the next to enter service




- FCFS is a special case where all have the same rate
- Expected waiting times can be calculated recursively for the multiclass (discrete) case (Kleinrock 76')
- LSTs derived in (Stanford, Taylor, and Ziedins 13')

# Strategic bidding

- M/G/1 queue
  - Arrival rate $\lambda$
  - Service first and second moments $\overline{x}$ and $\overline{x^2}$
  - Utilization $\rho = \lambda \overline{x}$
  - Steady state remaining service time $W_0 = \lambda \overline{x^2}/2$
  - Queueing time under FCFS $\frac{W_0}{1-\rho}$
- Unobservable queue
- Customers decide which AP rate to purchase out of a menu
- Cost of rate $b$ is $C_p b$
- Homogeneous linear waiting costs of $C_w$ per unit time
- One's waiting time also depends on the decisions made by others
- $\implies$ non-cooperative game

## Strategic bidding - equilibrium analysis

- A pure strategy is one of the menu's AP rate
- A mixed strategy is a distribution over the menu's AP rates
- A symmetric equilibrium strategy is such that if it is being used by everyone else, it is also your best response
- How should you react when more bid for higher priority?
  - you will be more likely to be overtaken $\implies$ bid higher
  - you will overtake less customers $\implies$ bid lower
- A priory unclear if to follow-the-crowd (FTC) or to avoid-the-crowd (ATC)

## The continuous menu case Haviv and Ravner 16'

- Customers can choose any priority $b \geq 0$
- The continuous version of Kleinrock's formula is developed
- Waiting time is shown to be strictly convex as a function of the individual rate for **any** given mixed strategy of the others
- A single pure best response strategy
- Implies that all equilibria are pure
- Direct analysis shows that the unique equilibrium is

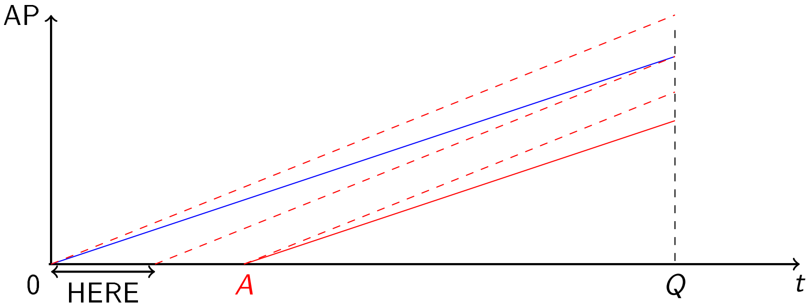
$$b_e = \frac{\rho W_0}{1-\rho}\frac{C_w}{C_p}$$

- Practically FCFS and all customers are worse off...
- How about the revenue? $C_p b_e$ is not a function of $C_p$

# The continuous menu case - an alternative approach

Abeywickrama et al 18'

- Everyone chooses rate $b$
- Look at the previous arrival
- This is what happens if you choose $b$ as well
- Nothing happens if you choose $b + \Delta$, $\Delta > 0$
- Unless you arrive earlier...



# The continuous menu case - an alternative approach

Abeywickrama et al 18'

- You will overtake if

$$Qb < (Q - A)(b + \Delta)$$

- Alternatively, if

$$A < Q\frac{\Delta}{b + \Delta}$$

- Happens with prob.

$$\lambda \mathbb{E}(Q)\frac{\Delta}{b + \Delta} + o(\Delta) = \lambda \frac{W_0}{1 - \rho}\frac{\Delta}{b + \Delta} + o(\Delta),$$

# The continuous menu case - an alternative approach

Abeywickrama et al 18'

- If it happens you overtake and save $\overline{x}C_w$
- But you had to pay extra $\Delta C_p$ anyways
- $\implies$ you should increase whenever

$$\left[\lambda \frac{W_0}{1-\rho} \frac{\Delta}{b+\Delta} + o(\Delta)\right] \overline{x}C_w > C_p \Delta$$

- Divide by $\Delta$, take the limit $\Delta \to 0$, and get

$$b < \frac{\rho W_0}{1-\rho} \frac{C_w}{C_p} = b_e$$

- Similar analysis for $\Delta < 0$ shows that you should decrease whenever

$$b > b_e$$

- Means that the only equilibrium is

$$b = b_e$$

## The discrete menu case Abeywickrama et al 18'

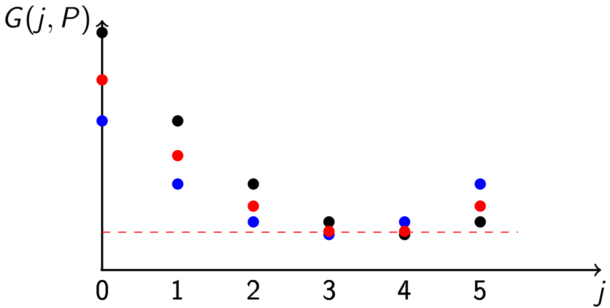
- Customers can only choose from a discrete menu, say, $\{0, 1, 2, \ldots\}$
- What will be the equilibrium now?
- $\lfloor b_e \rfloor$?
- $\lceil b_e \rceil$?
- Both?
- A mixture of both?

# Best response analysis

- $W(j, P)$ is the expected waiting of a customer who bids $j$ while the others use the mixed strategy $P = (p_0, p_1, \dots)$
- The total cost in that case is

$$G(j, P) = C_w W(j, P) + C_p j$$

- As said, $W(j, P)$ is strictly convex, so



# Equilibrium analysis

- Means that the best response set contains at most two consecutive pure strategies
- $\implies$ Two possible types of equilibria:
  - Pure
  - Mixed between two consecutive integers

## Pure equilibria

- Straightforward calculation shows that

$$W(i-1,i) - W(i,i) = \frac{W_0}{1-\rho}\frac{\rho}{i-\rho}$$

and

$$W(i,i) - W(i+1,i) = \frac{W_0}{1-\rho}\frac{\rho}{i+1}$$

- Means that $i$ is an equilibrium iff

$$\frac{\rho W_0}{1-\rho}\frac{1}{i-\rho} \geq C_p/C_w \geq \frac{\rho W_0}{1-\rho}\frac{1}{i+1}$$

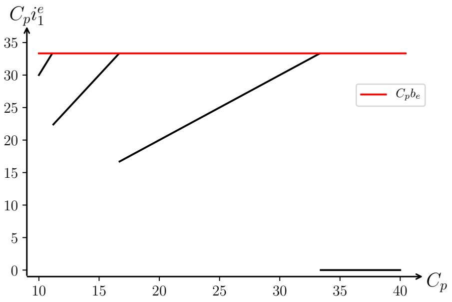
- Satisfied by $i_1^e = \lfloor b_e \rfloor$ and $i_2^e = \lfloor b_e + \rho \rfloor$
- If $i_1^e = i_2^e$ we have a unique pure equilibrium
- Otherwise we have two consecutive equilibria

# Revenue

- Equals

$$C_p \left\lfloor \frac{\rho W_0}{1-\rho} \frac{C_w}{C_p} \right\rfloor$$

- Locally insensitive to $C_w$
- Locally increasing with $C_p$
- However...



# Mixed equilibrium

- $(i+x)$, $0 \leq x \leq 1$, a real number denoting the strategy that mixes $i$ and $i+1$ w.p. $1-x$ and $x$
- $i+x$ is an equilibrium if both $i$ and $i+1$ are best response against it
- That is, if

$$W(i, i+x) - W(i+1, i+x) = \frac{W_0}{1-\rho} \frac{\rho}{i+1-x\rho} = C_p/C_w$$

- Possible only if $i_1^e \neq i_2^e$
- In that case it equals $i_1^e + x_e$ where

$$x^e = \frac{i_2^e - b_e}{\rho}$$

# Mixed equilibrium

- Work conservation law implies that

$$(1-x^e)W(i_1^e, i_1^e + x^e) + x^e W(i_1^e + 1, i_1^e + x^e) = \frac{W_0}{1-\rho}$$

- Calculations show that

$$W(i_1^e, i_1^e + x^e) = \frac{i_1^e + 1}{\rho} \frac{C_p}{C_w}$$

and

$$W(i_1^e + 1, i_1^e + x^e) = \left( \frac{i_1^e + 1}{\rho} - 1 \right) \frac{C_p}{C_w}$$

- A seeming paradox: both are (locally) decreasing in $\rho$ and $C_w/C_p$
- $x^e$ decreases with $C_w/C_p$ and $\rho$... Also counterintuitive

# Stability

- $P$ is an evolutionarily stable strategy (ESS) if for any other strategy $\tilde{P}$ either
  - $G(P, P) < G(\tilde{P}, P)$ or
  - $G(P, P) = G(\tilde{P}, P)$ and $G(P, \tilde{P}) < G(\tilde{P}, \tilde{P})$
- By definition, a unique equilibrium strategy is also an ESS
- When all three equilibria exist
  - The pure are ESS
  - The mixed is not (take $\tilde{P} = i_1^e$)
- A typical result in games with two pure equilibria and a third that mixes between the two

# Best response behavior

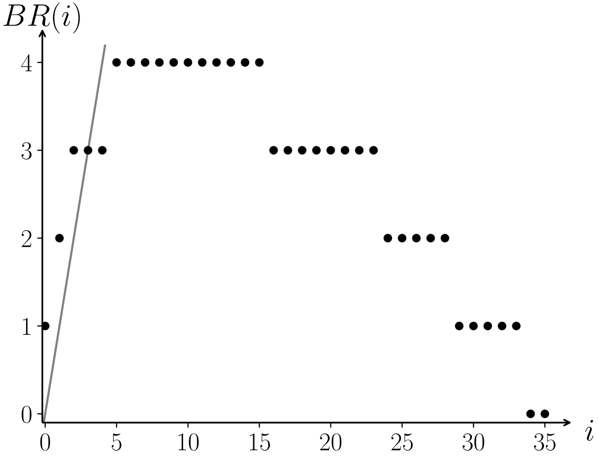
- $\mathcal{BR}(i)$ the best-response strategy against the pure strategy $i$

## Theorem

- *$\mathcal{BR}(i)$ is unimodal in $i$*
- *That is, we have FTC behavior for $0 \le i < i^*$ and ATC behavior for $i \ge i^*$, where $i^* = \max\{\arg\max_i \mathcal{BR}(i)\}$.*
- *The equilibrium strategies belong to the FTC part, i.e., $i_1^e \le i_2^e \le i^*$*

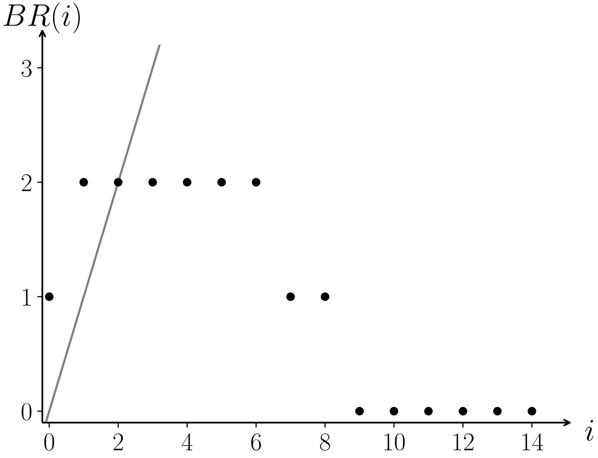
# Possible best response functions

Unique equilibrium in the increasing part



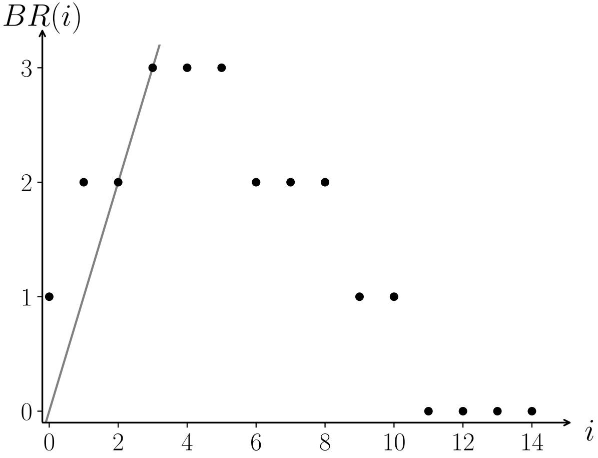
# Possible best response functions

Unique equilibrium in the plateau



# Possible best response functions

Two equilibria



THANK YOU