Strategic bidding in a discrete accumulating priority queue

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Accumulating priority queue

- Customer accumulate priority at a constant rate while in the system

- One with the greatest (accumulated) priority is the next to enter service

\[
\begin{align*}
\text{AP} & \quad \text{t} \\
0 & \quad 0
\end{align*}
\]

- FCFS is a special case where all have the same rate

- Expected waiting times can be calculated recursively for the multiclass (discrete) case (Kleinrock 76’)

- LSTs derived in (Stanford, Taylor, and Ziedins 13’)

Strategic bidding

- M/G/1 queue
  - Arrival rate $\lambda$
  - Service first and second moments $\bar{x}$ and $\bar{x}^2$
  - Utilization $\rho = \lambda \bar{x}$
  - Steady state remaining service time $W_0 = \lambda \bar{x}^2 / 2$
  - Queueing time under FCFS $\frac{W_0}{1-\rho}$

- Unobservable queue
- Customers decide which AP rate to purchase out of a menu
- Cost of rate $b$ is $C_p b$
- Homogeneous linear waiting costs of $C_w$ per unit time
- One’s waiting time also depends on the decisions made by others
- $\Rightarrow$ non-cooperative game
Strategic bidding - equilibrium analysis

- A pure strategy is one of the menu’s AP rate
- A mixed strategy is a distribution over the menu’s AP rates
- A symmetric equilibrium strategy is such that if it is being used by everyone else, it is also your best response
- How should you react when more bid for higher priority?
  - you will be more likely to be overtaken $\implies$ bid higher
  - you will overtake less customers $\implies$ bid lower
- A priory unclear if to follow-the-crowd (FTC) or to avoid-the-crowd (ATC)
The continuous menu case Haviv and Ravner 16’

- Customers can choose any priority $b \geq 0$
- The continuous version of Kleinrock’s formula is developed
- Waiting time is shown to be strictly convex as a function of the individual rate for any given mixed strategy of the others
- A single pure best response strategy
- Implies that all equilibria are pure
- Direct analysis shows that the unique equilibrium is

$$b_e = \frac{\rho W_0}{1 - \rho} \frac{C_w}{C_p}$$

- Practically FCFS and all customers are worse off...
- How about the revenue? $C_p b_e$ is not a function of $C_p$
The continuous menu case - an alternative approach

Abeywickrama et al 18’

- Everyone chooses rate $b$
- Look at the previous arrival
- This is what happens if you choose $b$ as well
- Nothing happens if you choose $b + \Delta$, $\Delta > 0$
- Unless you arrive earlier...

\[ \text{AP} \]

\[ \text{HERE} \]

\[ \text{A} \]

\[ \text{Q} \]

\[ t \]
The continuous menu case - an alternative approach

Abeywickrama et al 18’

• You will overtake if

\[ Qb < (Q - A)(b + \Delta) \]

• Alternatively, if

\[ A < Q \frac{\Delta}{b + \Delta} \]

• Happens with prob.

\[ \lambda \mathbb{E}(Q) \frac{\Delta}{b + \Delta} + o(\Delta) = \lambda \frac{W_0}{1 - \rho} \frac{\Delta}{b + \Delta} + o(\Delta), \]
The continuous menu case - an alternative approach

Abeywickrama et al 18’

• If it happens you overtake and save $\bar{x}C_w$
• But you had to pay extra $\Delta C_p$ anyways
• $\implies$ you should increase whenever

$$\left[ \lambda \frac{W_0}{1 - \rho} \frac{\Delta}{b + \Delta} + o(\Delta) \right] \bar{x}C_w > C_p\Delta$$

• Divide by $\Delta$, take the limit $\Delta \to 0$, and get

$$b < \frac{\rho W_0}{1 - \rho} \frac{C_w}{C_p} = b_e$$

• Similar analysis for $\Delta < 0$ shows that you should decrease whenever

$$b > b_e$$

• Means that the only equilibrium is

$$b = b_e$$
Customers can only choose from a discrete menu, say, \( \{0, 1, 2, \ldots \} \)

What will be the equilibrium now?

\[\left\lfloor b_e \right\rfloor? \]

\[\left\lceil b_e \right\rceil? \]

Both?

A mixture of both?
Best response analysis

- $W(j, P)$ is the expected waiting of a customer who bids $j$ while the others use the mixed strategy $P = (p_0, p_1, \ldots)$
- The total cost in that case is
  
  \[ G(j, P) = C_w W(j, P) + C_p j \]
  
- As said, $W(j, P)$ is strictly convex, so
Equilibrium analysis

- Means that the best response set contains at most two consecutive pure strategies
- $\implies$ Two possible types of equilibria:
  - Pure
  - Mixed between two consecutive integers
Pure equilibria

- Straightforward calculation shows that

\[ W(i - 1, i) - W(i, i) = \frac{W_0 \rho}{1 - \rho (i - \rho)} \]

and

\[ W(i, i) - W(i + 1, i) = \frac{W_0 \rho}{1 - \rho (i + 1)} \]

- Means that \( i \) is an equilibrium iff

\[ \frac{\rho W_0}{1 - \rho i - \rho} \geq \frac{C_p}{C_w} \geq \frac{\rho W_0}{1 - \rho i + 1} \]

- Satisfied by \( i_1^e = \lfloor b_e \rfloor \) and \( i_2^e = \lfloor b_e + \rho \rfloor \)

- If \( i_1^e = i_2^e \) we have a unique pure equilibrium

- Otherwise we have two consecutive equilibria
Revenue

- Equals
  \[ C_p \left[ \frac{\rho W_0}{1 - \rho \frac{C_w}{C_p}} \right] \]

- Locally insensitive to \( C_w \)
- Locally increasing with \( C_p \)
- However…

![Graph showing the relationship between \( C_p \) and \( C_{pie} \)]
Mixed equilibrium

• \((i + x), 0 \leq x \leq 1\), a real number denoting the strategy that mixes \(i\) and \(i + 1\) w.p. \(1 - x\) and \(x\)

• \(i + x\) is an equilibrium if both \(i\) and \(i + 1\) are best response against it

• That is, if

\[ W(i, i + x) - W(i + 1, i + x) = \frac{W_0}{1 - \rho} \frac{\rho}{i + 1 - x\rho} = C_p/C_w \]

• Possible only if \(i_1^e \neq i_2^e\)

• In that case it equals \(i_1^e + x_e\) where

\[ x_e = \frac{i_2^e - b_e}{\rho} \]
Mixed equilibrium

- Work conservation law implies that

\[ (1 - x^e)W(i_1^e, i_1^e + x^e) + x^e W(i_1^e + 1, i_1^e + x^e) = \frac{W_0}{1 - \rho} \]

- Calculations show that

\[ W(i_1^e, i_1^e + x^e) = \frac{i_1^e + 1}{\rho} \frac{C_p}{C_w} \]

and

\[ W(i_1^e + 1, i_1^e + x^e) = \left( \frac{i_1^e + 1}{\rho} - 1 \right) \frac{C_p}{C_w} \]

- A seeming paradox: both are (locally) decreasing in \( \rho \) and \( C_w/C_p \)

- \( x^e \) decreases with \( C_w/C_p \) and \( \rho \)... Also counterintuitive
Stability

- \( P \) is an evolutionarily stable strategy (ESS) if for any other strategy \( \tilde{P} \) either
  - \( G(P, P) < G(\tilde{P}, P) \) or
  - \( G(P, P) = G(\tilde{P}, P) \) and \( G(P, \tilde{P}) < G(\tilde{P}, \tilde{P}) \)

- By definition, a unique equilibrium strategy is also an ESS
- When all three equilibria exist
  - The pure are ESS
  - The mixed is not (take \( \tilde{P} = i_1^e \))

- A typical result in games with two pure equilibria and a third that mixes between the two
Best response behavior

- $BR(i)$ the best-response strategy against the pure strategy $i$

**Theorem**

- $BR(i)$ is unimodal in $i$
- That is, we have FTC behavior for $0 \leq i < i^*$ and ATC behavior for $i \geq i^*$, where $i^* = \max\{\arg\max_i BR(i)\}$.
- The equilibrium strategies belong to the FTC part, i.e., $i^e_1 \leq i^e_2 \leq i^*$
Possible best response functions

Unique equilibrium in the increasing part

\[ BR(i) \]
Possible best response functions

Unique equilibrium in the plateau

$$BR(i)$$
Possible best response functions

Two equilibria

$BR(i)$
THANK YOU