THE GHOST MCMC ALGORITHM:
rare event sampling and applications to power systems reliability

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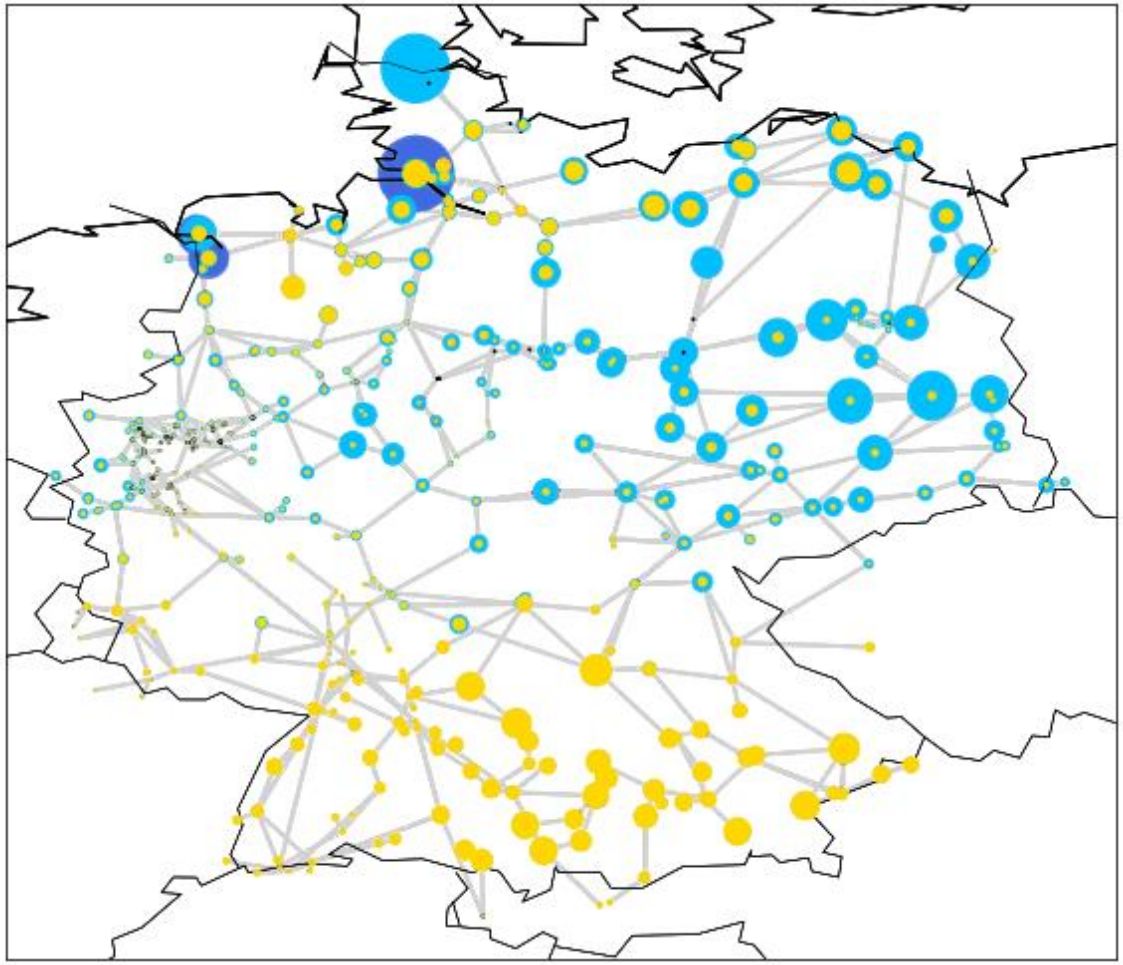
power grids are evolving rapidly

increasing penetration of renewables

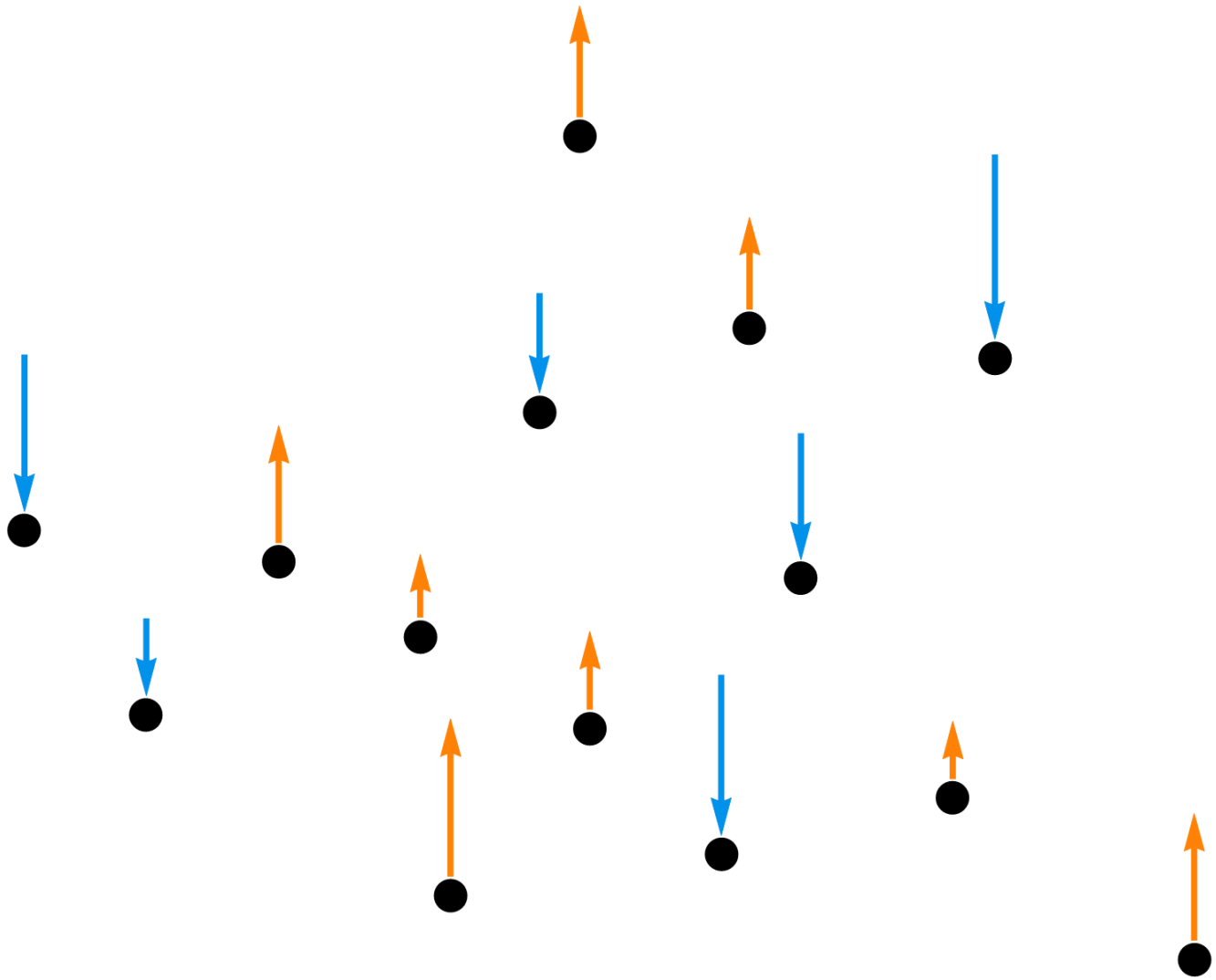
progressive transport electrification

can extreme fluctuations in load and generation cause failures?



German transmission network model with n = 585 nodes and m = 852 edges

renewables: blue = offshore wind, light blue = onshore wind, yellow = solar



Weighted graph with sources and sinks → power injections $\mathbf{p}$

power grids as stochastic network

power injections are modeled as *random variables*

$$\mathbb{E}\mathbf{p} = \mu \quad \text{and} \quad \mathrm{Cov}(\mathbf{p}) = \mathbf{\Sigma}_p$$

heterogeneous variances for the power injections fluctuations as well as correlations between them

→ non-trivial covariance matrix $\mathbf{\Sigma}_p$

quantities of interest

- line power flows $\mathbf{f} = V\mathbf{p}$
- nodal frequencies

$$M_j\dot{\omega}_j = -D_j\omega_j + (\mathbf{p}_j - \mu_j) - \sum_{i:(i,j)\in E} \mathbf{f}_{i,j}$$
$$\dot{\mathbf{f}}_{i,j} = B_{i,j}(\omega_i - \omega_j)$$

investigating power grids reliability

failure event = line power flow overloads

$$C^{\text{line}} = \bigcap_{\ell \in E} \{|\mathbf{f}_\ell| \geq 1\}$$

failure event = nodal frequencies or rocof violations

$$C^{\text{freq}} = \bigcap_{j \in \mathcal{G}} \left\{ \max_{t \in [0,T]} |\omega_j(t)| \geq \theta_{\max} \right\}$$

$$C^{\text{rocof}} = \bigcap_{j \in \mathcal{G}} \left\{ \max_{t \in [0,T]} |\dot{\omega}_j(t)| \geq r_{\max} \right\}$$

how does the system looks like upon failure?

how do failures most likely happen?

large deviations framework (NZZ18)

introduce the parameter $\varepsilon > 0$ to describe the magnitude of the system's noise

$$\mathrm{Cov}(\mathbf{p}) = \varepsilon \mathbf{\Sigma}_p$$

small-noise limit $\varepsilon \to 0^+$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{P}_\mu(|\mathbf{f}_\ell| \geq 1) = -I_\ell(\mu)$$

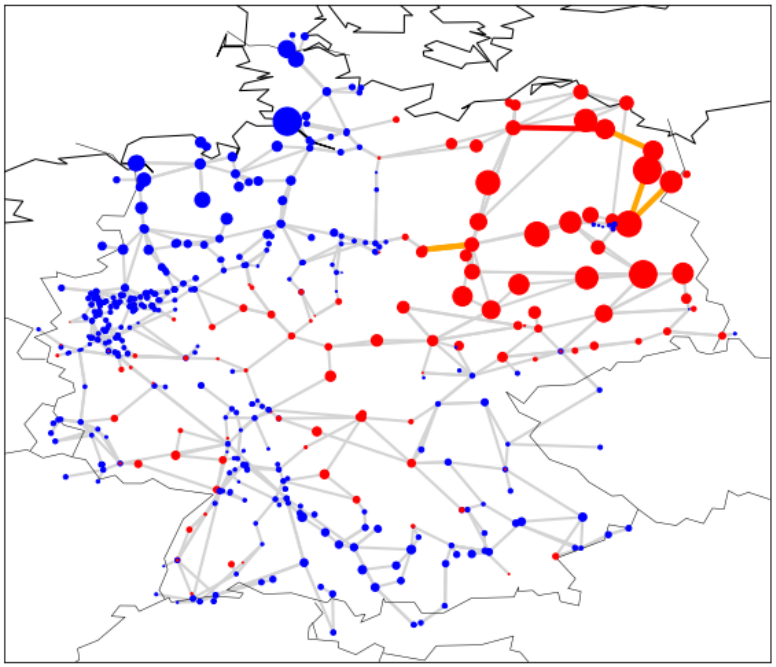
$$\lim_{\varepsilon \downarrow 0} \varepsilon \mathbb{P}_\mu(C^{\mathrm{line}}) = \lim_{\varepsilon \downarrow 0} \varepsilon \mathbb{P}_\mu(\max_\ell |\mathbf{f}_\ell| \geq 1) = -\min_\ell I_\ell(\mu)$$

in the Gaussian case

$$I_\ell(\mu) = \inf_{p \in \mathbb{R}^n : |e_\ell^T V p| \geq 1} \frac{1}{2}(p-\mu)^T \mathbf{\Sigma}_p^{-1}(p-\mu) = \frac{(1-|\nu_\ell|)^2}{2\sigma_\ell^2}$$

conditioning on failure event

$$\mathbf{p}^{(\ell)} = \underset{p \in \mathbb{R}^n \,:\, |\mathbf{e}_\ell^T V p| \geq 1}{\operatorname{arginf}} \frac{1}{2}(p-\mu)^T \mathbf{\Sigma}_p^{-1}(p-\mu) = \mathbb{E}[\,\mathbf{p} \,|\, \mathbf{f}_\ell = 1] = \mu + \frac{(1-\nu_\ell)}{\sigma_\ell^2} \mathbf{\Sigma}_p V^T \mathbf{e}_\ell$$

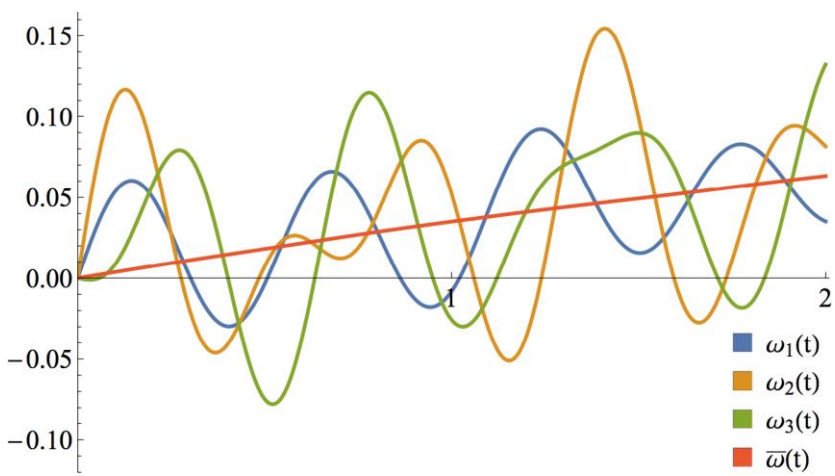


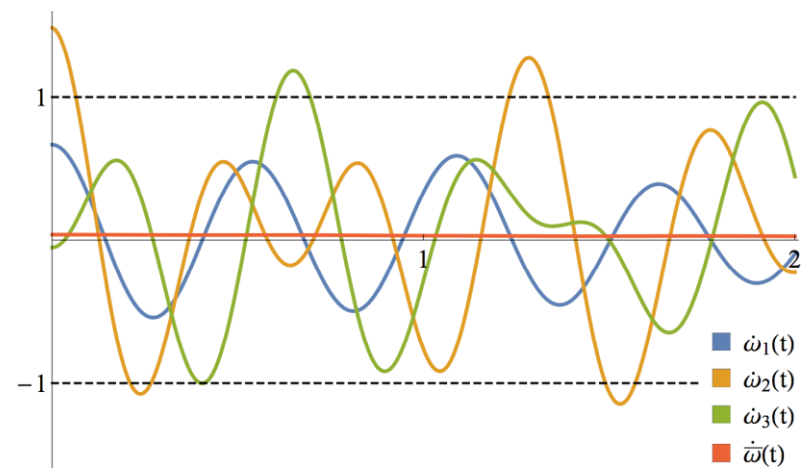
however, this close-form expression for power injections realization upon failure strongly depends on linearized dynamics and Gaussian assumption

how to deal with…?

- non-Gaussian & mixed distribution?
- non-linear dynamics?

$$\boxed{\begin{aligned} M_j\dot{\omega}_j &= -D_j\omega_j + (\mathbf{p}_j - \mu_j) - \sum_{i:(i,j)\in E} \mathbf{f}_{i,j} \\ \dot{\mathbf{f}}_{i,j} &= B_{i,j}(\omega_i - \omega_j) \end{aligned}}$$

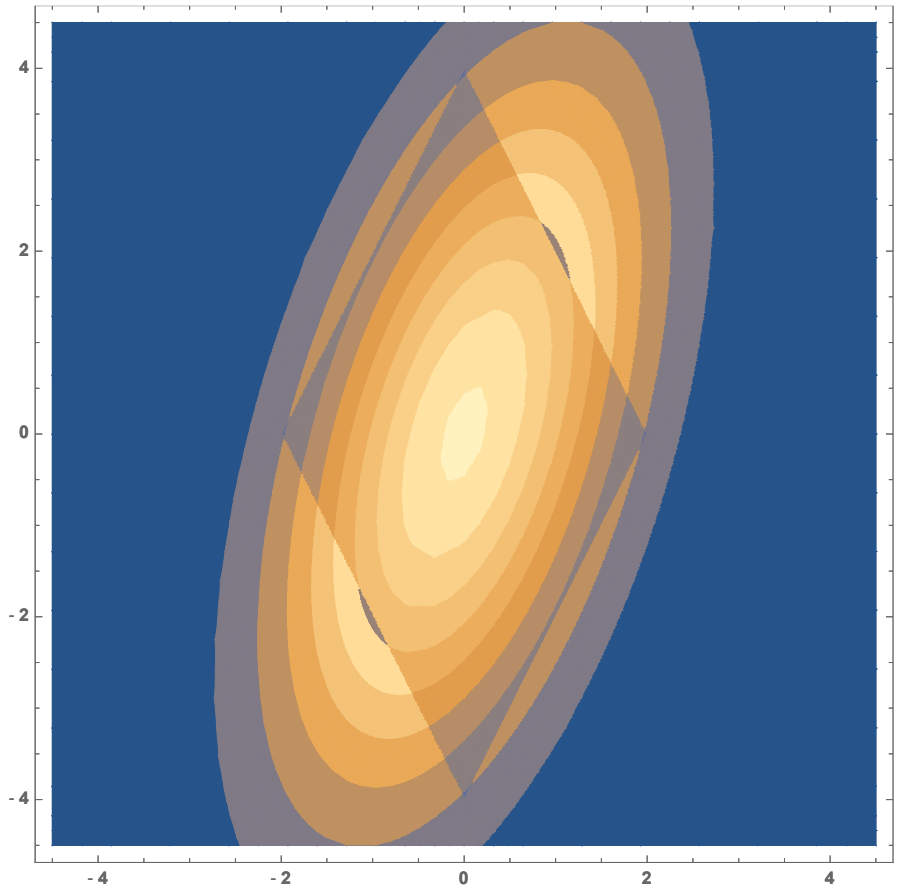




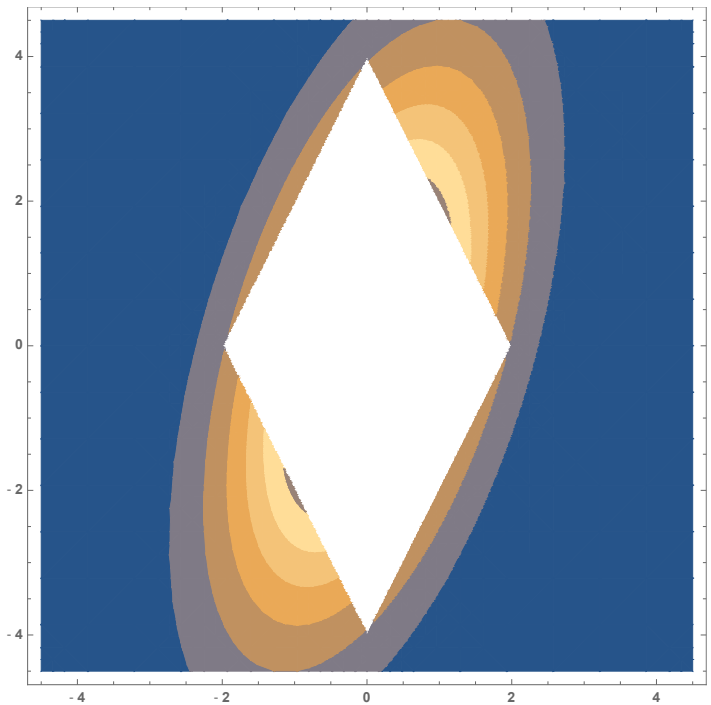
→ MCMC method to sample conditionally on failure event

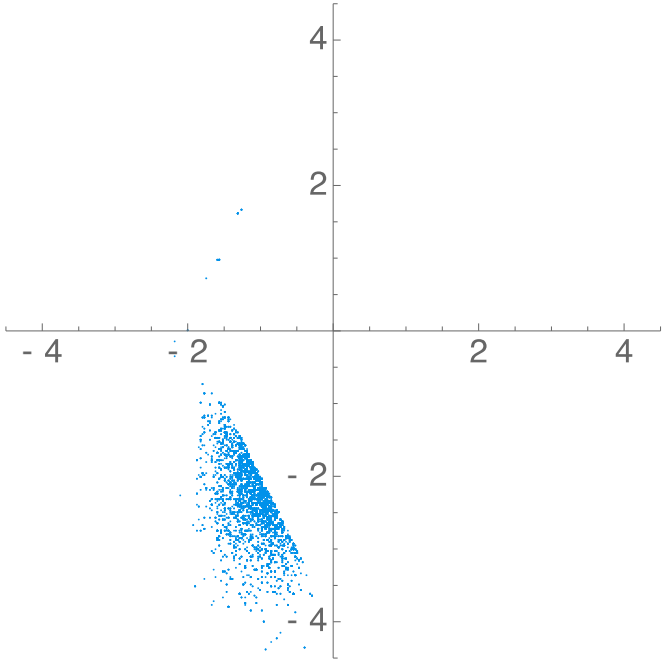
goal

devise a method to efficiently sample from the conditional distribution $\pi = \dfrac{\rho\, \mathbf{1}_C}{\rho(C)}$ where C is a rare event, i.e., $\rho(C) \ll 1$



how to sample?





Some naïve ideas…

- Sampling from ρ and rejecting when in C^c
- Sampling from π using MCMC

ghost algorithm

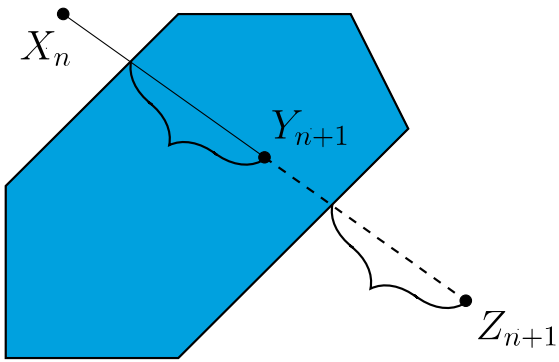
Algorithm 1: Ghost Random Walk Metropolis algorithm (n-th step)

Input : The n-th sample $X_n \in C \subset \mathbb{R}^d$

Generate a SRWM proposal Y_{n+1} distributed according to the density $q(y - X_n)dy$;
Calculate direction $\varphi_n = Y_{n+1} - X_n$;
Calculate all the intersection points (which are at most two) $T := \{t > 0 \,:\, X_n + t\varphi \in \delta C\}$;
if $\underline{T = \{t_1, t_2\} \textbf{ and } \min\{t_1, t_2\} < 1}$ **then**
  $Z_{n+1} = Y_{n+1} + (t_2 - t_1)\varphi$;
else
  $Z_{n+1} = Y_{n+1}$;
end
Evaluate the acceptance probability:

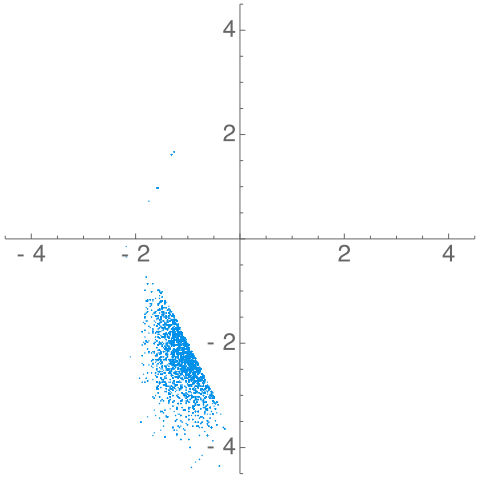
$$\alpha(X_n, Z_{n+1}) = \min\left(1, \frac{\pi(Z_{n+1})\mathbf{1}_C(Z_{n+1})}{\pi(X_n)\mathbf{1}_C(X_n)}\right)$$

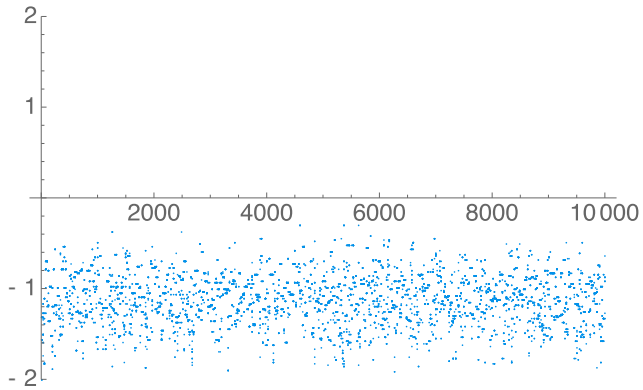
interpreted as one if $\pi(X_n)\mathbf{1}_C(X_n) = 0$:
Generate a uniform random variable U on $[0, 1]$;
if $\underline{U \leq \alpha(X_n, Z_{n+1})}$ **then**
  $X_{n+1} = Z_{n+1}$;
else
  $X_{n+1} = X_n$;
end
return $\underline{X_{n+1}}$

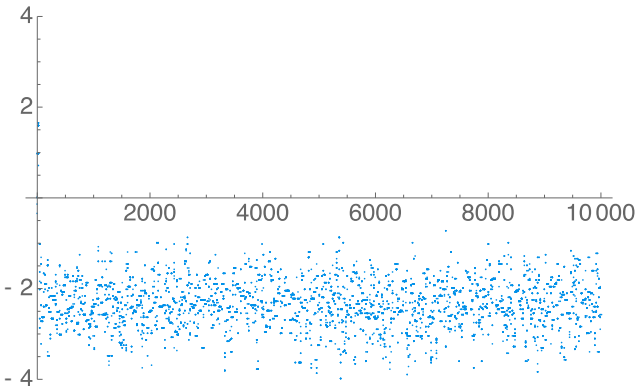


ghost algorithm in action

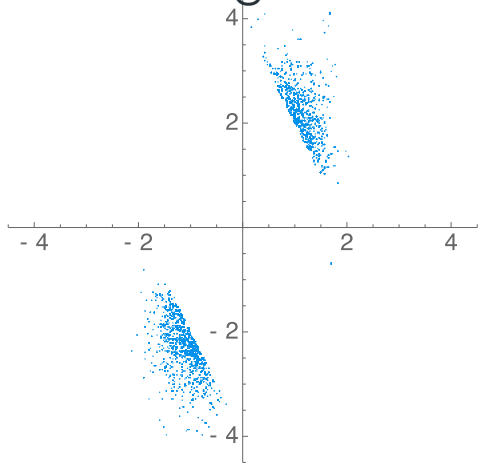
classical MCMC algorithm

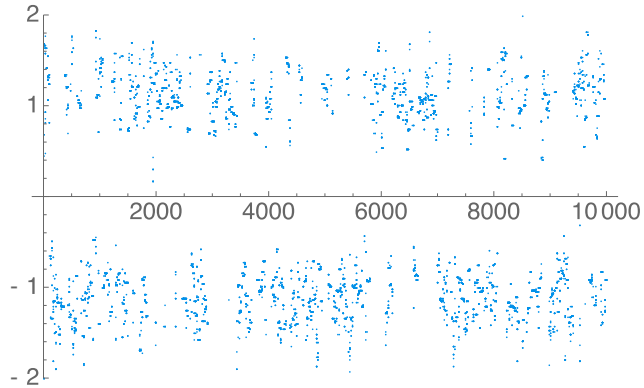


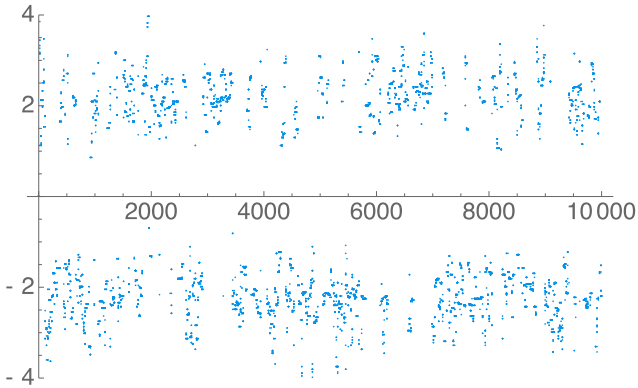




ghost MCMC algorithm







a technical condition (ray-boundedness)

define $l_x^{\varphi}(r) := \int_0^r \mathbf{1}_C(x + t\varphi)dt$ and $G_x^C : C \to \mathbb{R}^d$ as

$$G_x^C(x + r\varphi) := x + l_x^{\varphi}(r)\varphi$$

C closed set and $x \in C$ then $G_x^C : C \to \mathbb{R}^d$ injective

C^c **ray-bounded** if $G_x^C : C \to \mathbb{R}^d$ is surjective for all $x \in C$

C^c <u>convex polytope</u>: ray boundedness simply means that any ray starting in C and intersecting C^c will also exit from it

theorem (ghost sampling) (MV**Z**18)

Assume C is closed and C^c ray-bounded subset.

The ghost sampling RW is a MH algorithm with proposal density symmetric on C given by

$$q_{\mathrm{GS}}(x, x + r\varphi) := q(l_x^{\varphi}(r)\varphi) \left(\frac{l_x^{\varphi}(r)}{r} \right)^{d-1} \mathbf{1}_C(x + r\varphi).$$

Moreover, the GS algorithm is π- irreducible and the SLLN holds, that is, for every π- integrable function f

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} f(X_i) = \pi(f)$$

back to frequency violations

assume the system is affected by a step disturbance

$$u(t) = u\mathbf{1}_{\{t \geq 0\}}$$

where $u = p - \mu$ is a d-dimensional r.v. with distribution ρ

the system evolution is described by

$$\dot{x} = Ax \qquad x = \begin{bmatrix} \dot{\omega} \\ \omega \end{bmatrix}, \quad A = \begin{bmatrix} -M^{-1}D & -M^{-1}L \\ I & \mathbb{O} \end{bmatrix}, \quad x(0) = \begin{bmatrix} M^{-1}u \\ 0 \end{bmatrix}$$

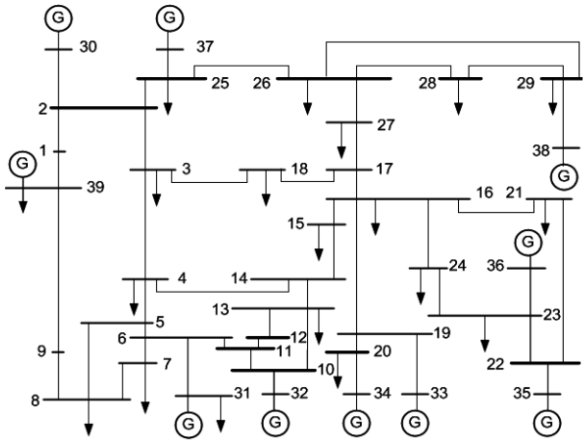
we approximate the event «rocof violation» in the interval $[0, \varepsilon]$ with $\varepsilon = 0.5\text{sec}$ as

$$C^{\text{rocof}}(N) = \bigcap_{j \in \mathcal{G}} \bigcap_{n=0}^{N} \left\{ u \in \mathbb{R}^d : \left| \dot{\omega}_j \left(\frac{n}{N} \varepsilon \right) \right| \leq r_{\max} \right\}$$

$$= \bigcap_{j \in \mathcal{G}} \bigcap_{n=0}^{N} \left\{ u \in \mathbb{R}^d : \left| \exp\left(\frac{n}{N} \varepsilon \right)_j \begin{bmatrix} M^{-1}u \\ 0 \end{bmatrix} \right| \leq r_{\max} \right\}$$

a case study

IEEE 39-bus test network



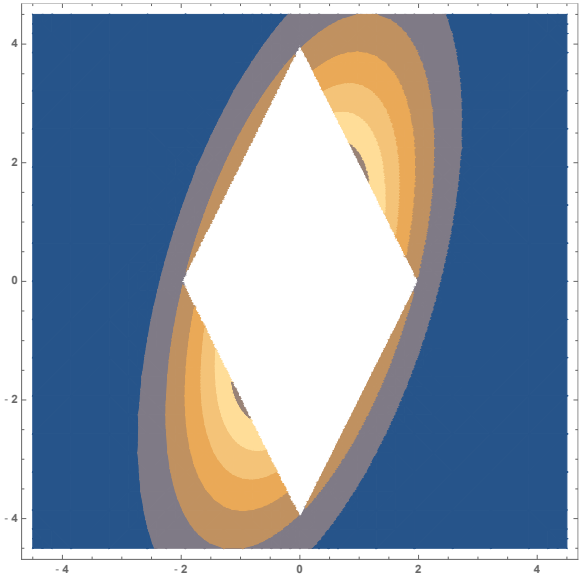
$$\rho(u_1, u_2) \sim \frac{1}{1 + (30(u_1 - u_2/2))^4} \cdot \frac{1}{1 + (30(u_2 - u_1/2))^4}$$

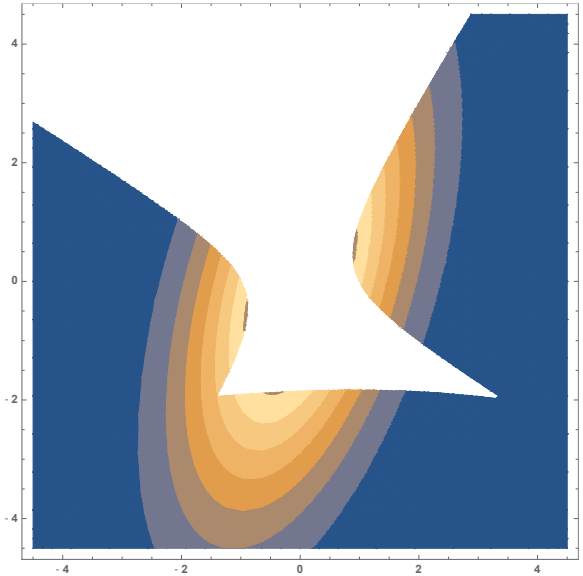
+ the other components i.i.d. zero-mean Gaussian

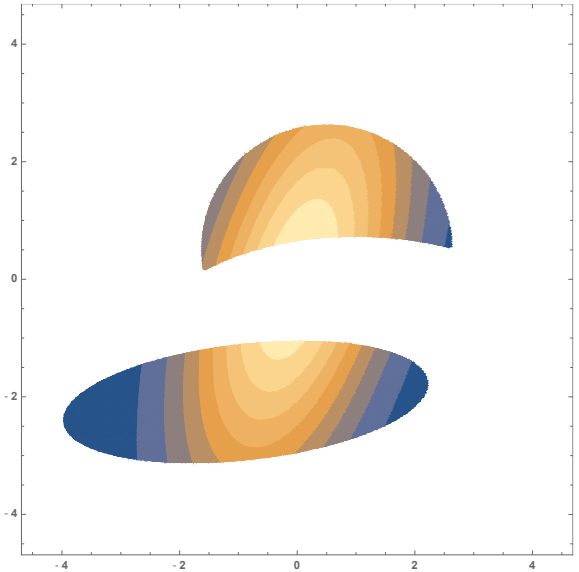
N	$G1$	$G2$	$G3$	$G4$	$G5$	$G6$	$G7$	$G8$	$G9$
1	28.9	80.3	0.5	0.6	0.9	0.4	1.6	6.5	1.6
5	27.6	81.5	12.4	1.1	2.1	0	1.2	9.8	1.7
20	27.5	79.5	11.5	1.9	3.0	0.1	2.4	15.5	2.0
50	28.5	78.8	12.2	1.1	2.7	0.1	2.4	17.1	2.6
100	28.6	79.8	12.2	1.7	2.4	0.1	1.9	15.6	2.0

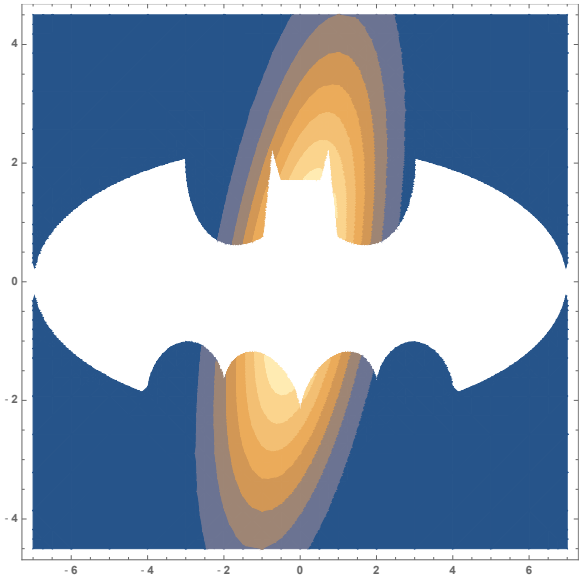
	$N = 1$	$N = 5$	$N = 20$	$N = 50$	$N = 100$
p_d	15.2%	22.4%	24.0%	25.0%	25.0%
$\overline{d}$	1.21	1.37	1.44	1.46	1.44
$\overline{L}$	596	701	735	744	736

extension to more general events









some preliminaries

$\mathcal{K} = \{K_\varphi \mid \varphi \in \mathbb{S}^{d-1}\}$ is a **halting regime** if

- for every $\varphi \in \mathbb{S}^{d-1}$ K_φ is a r.v. on $\mathbb{N} \cup \{\infty\}$
- $K_\varphi \sim K_{-\varphi}$ for every direction $\varphi \in \mathbb{S}^{d-1}$

assume the proposal density can be written as

$$q(y-x) = q_{r|\varphi}(r|\varphi)q_\varphi(\varphi)$$

where $r = |y-x|$ and $\varphi = (y-x)/|y-x|$

if the proposal density is $q(x-y) \sim \mathcal{N}(\mathbf{0}, \Sigma)$, then

$$q_\varphi(\varphi) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2} \cdot \sqrt{\det\Sigma} \cdot (\varphi^T \Sigma^{-1} \varphi)^{d/2}}$$

$$q_{r|\varphi}(r) = \frac{(\varphi^T \Sigma^{-1} \varphi)^{d/2}}{2^{d/2-1}\Gamma(\frac{d}{2})} e^{-(\varphi^T \Sigma^{-1} \varphi)\frac{r^2}{2}} r^{d-1}$$

$q_{r|\varphi}(r)$ is **generalized gamma distribution** with zero location parameter, scale parameter

$$\beta = \left(\frac{2}{\varphi^T \Sigma^{-1} \varphi}\right)^{1/2}$$

and shape parameters

$$\alpha = d/2 \qquad \gamma = 2$$

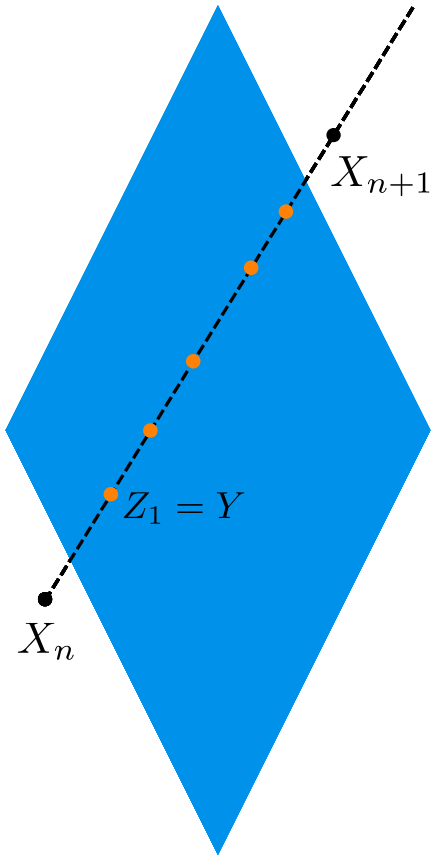
Algorithm 2: Skipping Random Walk Metropolis algorithm (n-th step)

Input : The n-th sample $X_n \in \mathbb{R}^d$

Set $X := X_n$;
Generate the initial SRWM proposal Y distributed according to the density $q(u - X)du$;
Calculate the direction $\phi = (Y - X)/|Y - X|$;
Generate a halting index $K \sim K_\phi$;
Set $k = 1$ and $Z_1 := Y$;
while $\underline{Z_k \in C^c \text{ and } k < K}$ **do**
  Generate a distance increment R distributed according to $q_{r|\varphi}(\cdot|\phi)$;
  Set $Z_{k+1} = Z_k + \phi R$;
  Increase k by one;
end
Set $Z := Z_k$;
Evaluate the acceptance probability:

$$\alpha(X, Z) = \begin{cases} \min\left(1, \frac{\pi(Z)}{\pi(X)}\right) & \text{if } \pi(X) \neq 0, \\ 1, & \text{otherwise,} \end{cases}$$

Generate a uniform random variable U on $(0, 1)$;
if $\underline{U \leq \alpha(X, Z)}$ **then**
  $X_{n+1} = Z$;
else
  $X_{n+1} = X$;
end
return $\underline{X_{n+1}}$



another technical condition

let T_C be the first entry time of the skipping chain into C, that is $T_C := \min\{k \geq 1 \ : \ Z_k \in C\}$.

assume the halting regime $\mathcal{K} = \{K_\varphi \mid \varphi \in \mathbb{S}^{d-1}\}$ and the subset C are such that

$$\forall\, x \in \mathbb{R}^d \qquad \sup_{\varphi \in \mathbb{S}^{d-1}} \mathbb{E}_x \left[T_C \wedge K_\varphi\right] < \infty$$

theorem (skipping sampling) (MV**Z**19**?**)

Assume the halting regime $\mathcal{K} = \{K_\varphi \mid \varphi \in \mathbb{S}^{d-1}\}$, the subset C and the proposal density are as before.

Then the skipping RWM algorithm:

- is a MH algorithm for a proposal density q_{SK} which is symmetric on C;
- is π- irreducible and has unique stationary probability measure π;
- the SLLN holds;
- is provably **faster** than the classical MH algorithm.

future work

- extensive numerics for power grids

  - large-scale transmission network
  - realistic distributions for disturbances

- finish writing the paper about the skipping sampler…

- find new applications for skipping sampling

  - queueing systems?
  - random matrices?
  - random graphs?

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MVZ18 J.Moriarty, J. Vogrinc, A. Zocca «Frequency violations from random disturbances: an MCMC approach», To appear in *2018 IEEE 57th Annual Conference on Decision and Control (CDC)*, arXiv:1803.08522

NZZ18 T. Nesti, A. Zocca, B. Zwart «Emergent failures and cascades in power grids: a statistical physics perspective», In *Physical Review Letters* 120, 258301

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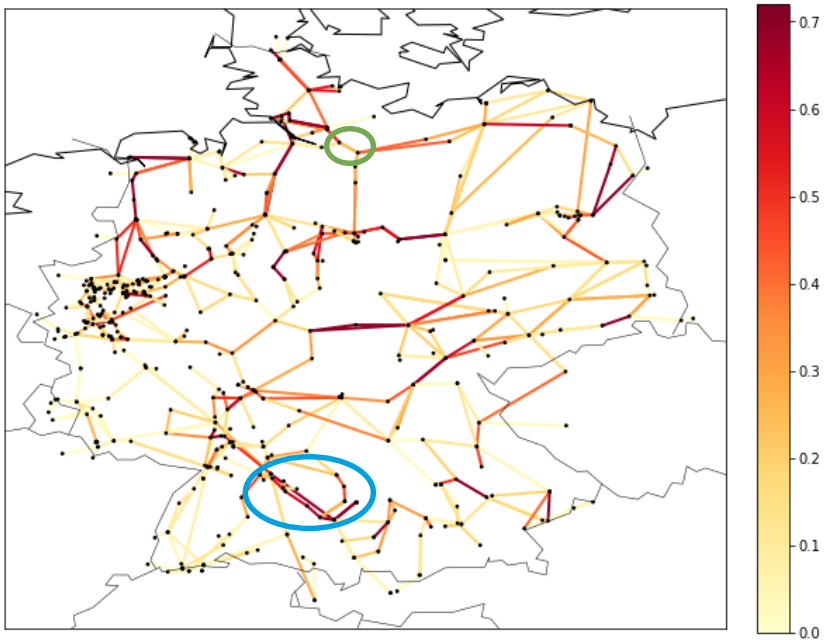
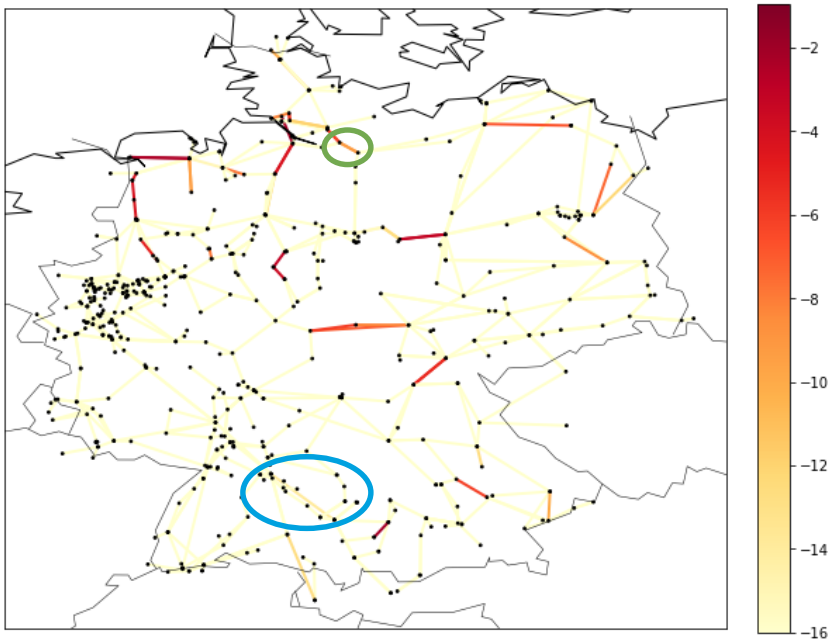
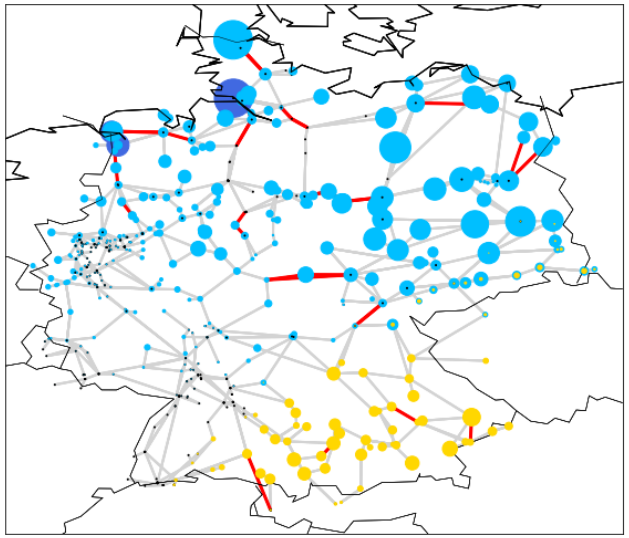


absolute values power flows

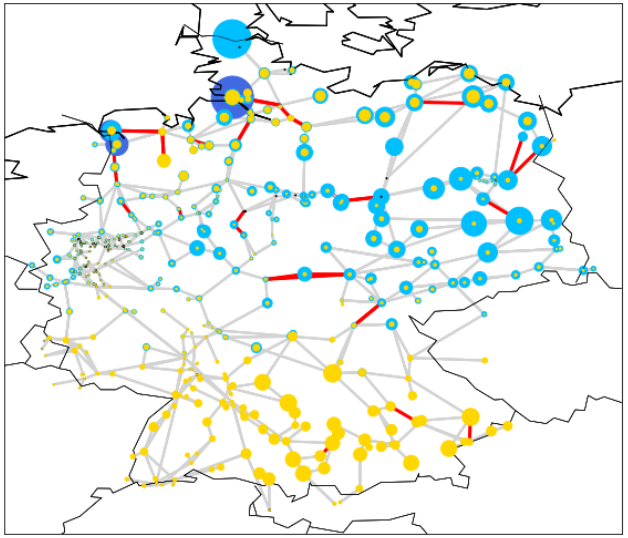


line failure decay rates

our large deviations framework enables the
correct identification of network vulnerabilities!



8am



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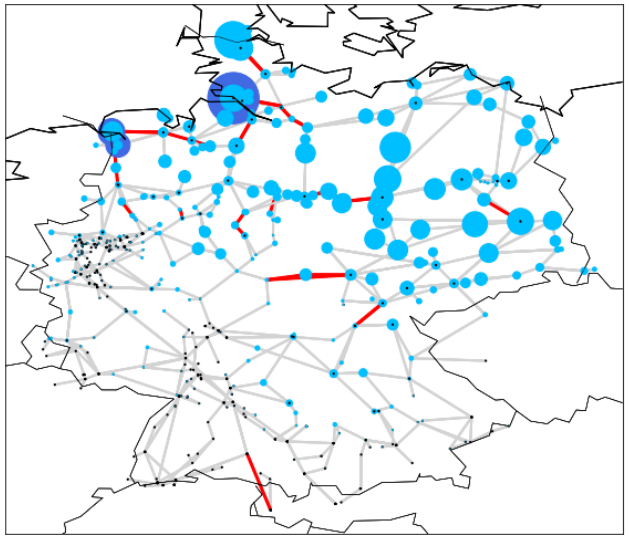

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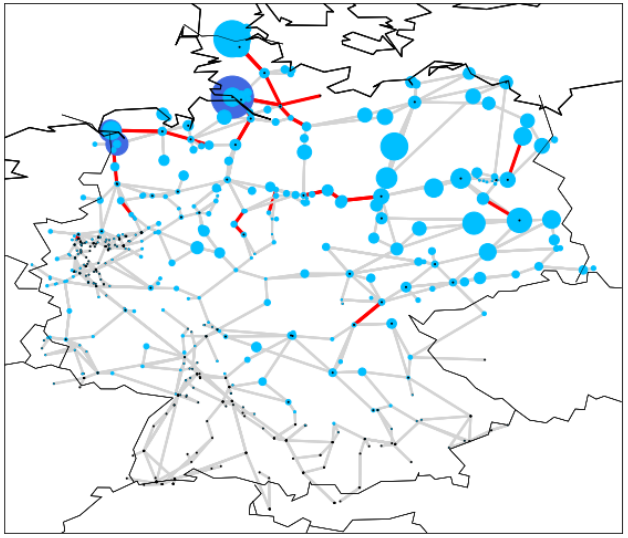

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